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BASIC PROPERITES OF NONSYMMETRIC UNIFIED FIELD THEORIES

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Basic Properties of Nonsymmetric Unified Field Theories" submitted by David Ronald Noakes in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics.

Abstract

The various types of unified field theory of gravitation and electromagnetism are reviewed, and that which is presently known about Bonnor-Moffat-Boal Theory is outlined in some detail. It is shown that there is no simple method to generalize the concept of duality of the electromagnetic field to nonsymmetric theories in a meaningful way. Several families of possible solutions important for analysing the viability of Bonnor-Moffat-Boal theory are described.

An attempt is made to physically interpret the geometric structure of Bonnor-Moffat-Boal theory (and by generalization, other theories) to the extent that general relativity's geometric structure has found interpretation. Many difficulties threatening the viability of the theory are found and left unresolved.

A weak perturbation expansion formalism is developed and with it an approximate magnetic monopole solution is found in the theory. The geometric optics approximation is developed and applied to the theory, to study the propagation of light and gravitational radiation. In relatively simple situations radiation is found to travel along null geodesics. In more complicated circumstances, difficulties are encountered, and no conclusions can immediately be drawn regarding propagation of radiation in these cases.

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Review of Unified Field Theories

Introduction

This thesis is mainly concerned with gaining a deeper understanding of Bonnor-Moffat-Boal unified field theory (BMB theory, hereafter) of gravitation and electromagnetism. In the latter parts of this chapter we will review other unified field theories to gain an overall view of the subject and to be able to compare and contrast BMB theory with its competitors, which we will do in subsequent chapters. In chapter 2 we will go on to review BMB theory as it is presently known, and consider some special circumstances in which new solutions of the equations might (be naively expected to) exist. In chapter 3 we will consider just how much of general relativity's deep geometric structure can be found in unified field theories with nonsymmetric metrics, BMB theory in particular. In chapter 4 we will develop the formalism of weak perturbation expansions about exact solutions, primarily as a step toward investigating "geometric optics" in chapter 5, but with some obvious applications of its own. In chapter 5 we will investigate light and gravitational radiation within the framework of the geometric optics approximation, mainly for the purpose of deciding whether such radiation travels on null geodesics or not. In chapter 6 we draw such conclusions as we can from the work done.

Weyl's Theory

Einstein's general relativity (which we will also call Einstein-Maxwell Theory and abbreviate EMT, when we are considering gravity plus electromagnetism) couches all quantities relating to the gravitational field in terms of geometric objects on manifolds. The gravitational effects of other fields are included by identifying their stress-energy-momentum tensor T_{ab} with the Einstein tensor G_{ab}:

$$G_{ab} = 8\pi T_{ab}. \tag{1.1}$$

G_{ab} is a properly geometric gravitational quantitly, but T_{ab} has no <u>a priori</u> geometric content whatsoever. Thus the other fields do not enter in a geometric way.

The only other field known at the time general relativity was first proposed (it is still the best understood field) was the electromagnetic field. Several researchers including Einstein soon began to look for a way to combine gravitation and electromagnetism in a "natural" geometric way, so that both emerge from geometry on manifolds. Further motivation for this was provided when it was found that exact solutions in EMT consistently display singularities,locations where the curvature quantities diverge. It was thought that a correct theory should generate elementary particles as regular solutions, and that modifying EMT to a correct "unified field theory" might get rid of the singularities. So far, this has been a false hope[1].

The first such unified field theory was Weyl's (1918)[2,3]. (For review of Weyl's theory see refs. 4,5). In it he proposed what today would be called a gauge theory of length on a manifold with metric. In general relativity the length of a vector is preserved in parallel transport. That is, the inner product of two vectors:

$$k^a j_a = g_{ab} k^a j^b \tag{1.2}$$

of which length is a special case:

$$l^2 = j^a j_a = g_{ab} j^a j^b \quad , \tag{1.3}$$

is not changed when we move the vector from point to point according to:

$$dj^a = \Gamma^a_{bc} j^b dx^c \quad . \tag{1.4}$$

In more modern notation, for transport in the direction of k^a, this is expressed as:

$$k^b j_{a;b} = 0 \tag{1.5}$$

(semicolon denotes covariant differentiation with respect to the connection Γ^a_{bc}). This is so because we arrange that the metric be compatible with the connection:

$$g_{ab;c} = 0 \quad . \tag{1.6}$$

Weyl suggested[6] letting the length vary with displacement, and as a first relaxation of invariant length we let the variation be linear in length for infinitesmal displacement:

$$dl = (\phi_a dx^a)l. \tag{1.7}$$

Note that the new vector quantity ϕ_a enters (1.7) in the same manner that the connection enters (1.4).

From:

$$dl^2 = 2l^2\phi_a dx^a = d(g_{ab}j^a j^b) \tag{1.8}$$

we can quickly show:

$$\Gamma^a_{bc} = \Gamma^a_{bc(lc)} - g^{da}(g_{db}\phi_c + g_{dc}\phi_b - g_{bc}\phi_d) \tag{1.9}$$

where $\Gamma^a_{bc(lc)}$ is the standard Levi-Civita connection found in EMT. Note that both g_{ab} and ϕ_a are needed to determine Γ^a_{bc}.

Now, consider conformally rescaling the metric with some scalar field $f(x^a)$:

$$\hat{g}_{ab} = fg_{ab} \qquad (f > 0) \quad . \tag{1.10}$$

This modifies lengths:

$$\hat{l}^2 = fl^2 \tag{1.11}$$

as if we had a new ϕ_a:

$$\hat{\phi}_a = \phi_a + \frac{\frac{1}{2}f_{,a}}{f} = \phi_a + \tfrac{1}{2}(\ln f)_{,a} \tag{1.12}$$

$$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc} \tag{1.13}$$

(where the comma refers to coordinate differentiation). The Weyl geometry thus admits a "gauge" transformation, and we call ϕ_a the gauge vector field. Quantities which do not change under such transformations,

gauge invariants, will be considered of greater physical significance than gauge dependent quantities.

If ϕ_a is zero, we regain length preservation, and are in the realm of general relativity again(7). We can reduce ϕ_a to zero by a gauge transformation if it is the gradient of a scalar, which is equivalent to:

$$F_{ab} = \phi_{a,b} - \phi_{b,a} = 0. \qquad (1.14)$$

F_{ab} is a gauge invariant antisymmetric tensor of the second rank. By construction it satisfies:

$$F_{[ab,c]} = 0. \qquad (1.14)$$

The brackets here indicate antisymmetrization over all three indices.

We can still identify geodesics using parallel transport:

$$k^b k_{a;b} = 0 \qquad (1.16)$$

but these geodesics are not extremals of arc length, since nonzero arc lengths are now gauge dependent. Null vectors and geodesics, however, are still invariant. We can form curvature tensors in the usual way. As an example, the curvature scalar has the form:

$$R = R_{emt} - 6\,\phi_a \phi^a + \frac{6}{\sqrt{-g}}\left(\sqrt{-g}\,\phi^a\right)_{,a} \qquad (1.17)$$

(R_{emt} is the curvature scalar as would be found in EMT, g is the determinant of the metric).

Let us label an entity by the way it gauge transforms. If $T^{a\ldots}{}_{b\ldots}$ transforms as:

$$\hat{T}^{a\ldots}{}_{b\ldots} = f^n T^{a\ldots}{}_{b\ldots} \qquad (1.18)$$

then $T^{a\ldots}{}_{b\ldots}$ is said to have a weight of n under the gauge transformation. Note that raising and lowering indices changes weight. Some examples of these that will concern us are given in Table 1. In it :

$$\underline{F}^{ab} = \sqrt{-g}\, F^{ab}; \qquad (1.19)$$

The underline indicates a tensor density.

TABLE 1

Object	Weight
$\sqrt{-g}$	2
F^{ab}	-2
$\underline{F}^{ab}$	0
$R^{a}{}_{bcd}$	0
R^{ab}	0
R	-1

A conventional way to find field equations is to use a variational principle:

$$L = \delta\int W\sqrt{-g}\,d^4x = 0 \tag{1.20}$$

(g_{ab} and ϕ_a will be varied independently), in which $W\sqrt{-g}$ should be a gauge invariant scalar density. On purely formal grounds:

$$\int(\underline{W}^a\,\delta\phi_a + \underline{W}^{ab}\,\delta g_{ab})\,d^4x = 0 \tag{1.21}$$

implies:

$$\underline{W}^a = 0 \qquad , \qquad \underline{W}^{ab} = 0 \quad . \tag{1.22}$$

These are fourteen equations, but they do not comprise an independent set. We can generate identities by formal gauge and coordinate variations in (1.20). An infinitesmal gauge transformation implies:

$$\underline{W}^a{}_{,a} = 2\underline{W}^a{}_a \ . \tag{1.23}$$

Infinitesmal coordinate transfomations generate a further four identities. Thus (1.22) really represents nine independent field equations.

The theory begins to show weakness when we look for the particular form of W. It is reasonable to ask that W be constructed from the metric, its first and second derivatives, the gauge vector and its first derivatives. These restrictions give us only four rational functions:

$$F_{ab}F^{ab} \ , \ R_{abcd}R^{abcd} \ , \ R_{ab}R^{ab} \ , \ R^2 \ . \tag{1.24}$$

It was hoped at the time of proposal that the correct unified field theory would have its Lagrangian uniquely determined with no unknown factors (the values of which must be guessed) and with the gravitational and electromagnetic terms appearing together, not artificially separated. Here we have the undesirable situation of four expressions to be combined somehow, including one ($F_{ab}F^{ab}$) which is strictly electromagnetic. Further, there is no way to construct a generalization of the Lagrangian of EMT:

$$R + 4\, F_{ab}F^{ab} \tag{1.25}$$

because now R is not gauge invariant. The closest we can get to this is:

$$W = -\tfrac{1}{4}\, R^2 + A\, F_{ab}F^{ab} \tag{1.26}$$

and this is therefore the expression considered most likely to succeed.

An immediate problem is: this will, as it stands, generate gravitational field equations of fourth order, instead of EMT's second order. Weyl tried to get around this by a particular choice of gauge.[8]

$$0 = \delta\int W\sqrt{-g}\,d^4x = \int\left[-\tfrac{1}{2}R(\delta R)\sqrt{-g} + \tfrac{1}{4}R^2\delta\sqrt{-g} + A\delta(F_{ab}F^{ab}\sqrt{-g})\right]d^4x. \tag{1.27}$$

Since R is of weight one (and in cases for which R is of single sign) we can choose a gauge so that:

$$R = -1 = \text{constant} \ . \tag{1.28}$$

This is a measure of the overall curvature of the spacetime, somewhat in the sense of the cosmological constant. We can then bring our varied action into a more recognizable form:

$$\delta\int\left[R_{emt} + AF_{ab}F^{ab} - \frac{3}{4}(\phi_a\phi^a) + \tfrac{1}{4}\right]\sqrt{-g}\,d^4x = 0 \ . \tag{1.29}$$

The normalization used implies cosmic-sized units. To go to conventional units:

$$\tilde{x}^a = \varepsilon x^a \qquad , \qquad \varepsilon \ll 1 \tag{1.30}$$

with simultaneous rescaling:

$$\tilde{g}_{ab} = \frac{1}{\varepsilon^2}\, g_{ab} \tag{1.31}$$

$$\tilde{\phi}_a = \phi_a \qquad , \qquad \tilde{F}^{ab} = F^{ab} \tag{1.32}$$

$$\delta\int\left[R_{emt} + A\tilde{F}_{ab}\tilde{F}^{ab} + \frac{\varepsilon^2}{4}(1 - 3\tilde{\phi}_a\tilde{\phi}^a)\right]\sqrt{-g}\,d^4x = 0 \ . \tag{1.33}$$

This looks like the vacuum EMT expression with correction terms on the end which we hope are small because of the smallness of the cosmological term ε^2. To generate Einstein's cosmological constant:

$$\tfrac{1}{2}\lambda\sqrt{-g} \tag{1.34}$$

we set:

$$\lambda = \frac{\varepsilon^2}{2} \; . \tag{1.35}$$

By varying ϕ_a we get electromagnetic equations:

$$\begin{aligned} \tilde{\underline{F}}^{ab}{}_{,b} &= -\frac{3}{\underline{A}}\lambda\tilde{\phi}^a\sqrt{-g} \\ &= \tilde{\underline{s}}^a \end{aligned} \tag{1.36}$$

and charge conservation is automatic:

$$\tilde{\underline{s}}^a{}_{,a} = 0 \; . \tag{1.37}$$

However, to get a gauge invariant expression for (1.36) we use the gauge invariant:

$$\phi_a + \tfrac{1}{2}\,(\ln R)_{,a} \quad , \tag{1.38}$$

implying:

$$\tilde{\underline{s}}^a = \frac{-3}{A\sqrt{\lambda}}\,(R\phi_b + \tfrac{1}{2}R_{,b})\; g^{ab}\sqrt{-g} \; . \tag{1.39}$$

Variation of the metric components gives:

$$R_{ab} - \frac{(R+\lambda)}{2}\, g_{ab} = AT_{ab} \tag{1.40}$$

$$T_{ab} = (F_{cd}F^{cd} + \tfrac{1}{2}\phi_c s^c)\; g_{ab} - F_{ac}F_b{}^c - \phi_a s_b \; . \tag{1.41}$$

Einstein[9] immediately raised an objection, based upon what we will call "geometric interpretation", that is, based upon the demand that all the physically interpretable geometric quantities in EMT should have analogies in unified field theory. In EMT, the arc length of a timelike path is related to the elapsed proper time, and the length of the tangent to the path is the rate at which the proper time passes. If we apply this directly to Weyl's unified field theory, we find that the rate at which a particular clock ticks depends upon the value of ϕ_a at the clock's position, and in general continually changes. We can easily show[10] then that two identical clocks at the same location with different past histories have different rates of ticking, the difference increasing with time. This past dependent behavior is surprising and is considered unlikely. An interesting reinterpretation

was provided by London[11], who showed that if we ask for geodesics along which this length variation does not occur (thus avoiding the above stated problem), we start to get quantum mechanical results, eg. Bohr's quantization in a spherically symmetric field. However, this idea was not extended far.

While the generalization of EMT that Weyl proposed is geometrically elegant, the Lagrangian for the field equations is not uniquely defined, and the resulting field equations do not seem to be a simple generalization of those found in EMT. Possibly because of the complexity of the field equations, Weyl's theory was soon abandoned.

<u>Five Dimensional Theories</u>

The five dimensional (including " projective") unified field theories form a family unrelated to Weyl's theory. This type was first proposed by Kaluza[12-14], in 1921, and later extended by Einstein and collaborators[15-17]. In the forthcoming analysis, we will follow Bergmann's[4] review.

These theories use five dimensional Riemannian geometry in which four dimensions are distinguished from the fifth. We will begin our review by considering a general "four dimensions within five" formalism.

In a five dimensional space with coordinates y^{α}, metric $g_{\alpha\beta}$ ("α" indicates five-space indices $0,\ldots 4$;"a" indicates four-space indices $0,\ldots,3$), we introduce four parameters x^a, functions of y^{α}, and assume their derivatives:

$$Q^a_{\alpha} = x^a{}_{,\alpha} \tag{1.42}$$

are linearly independent. We have a set of curves, $x^a = \text{const}$, which fills the five-space, and which forms a structure on the space invariant under the "parameter transformations":

$$\tilde{x}^a = f^a(x^b) \quad . \tag{1.43}$$

We can find a large number of geometric objects in this formalism. There are ordinary five dimensional tensors (indicated by Greek indices), and p-tensors, or parameter tensors (indicated by Latin indices), transforming as four-dimensional tensors with respect to parameter transformations. Entities with some Latin and some Greek indices mix the transfomation properties above. We identify a fundamental vectorfield A^{α} (tangent to the curves) by:

$$Q^a_{\alpha} A^{\alpha} = 0 \tag{1.44}$$

$$g_{\alpha\beta} A^{\alpha} A^{\beta} = 1 \tag{1.45}$$

and define a "reciprocal Q":

$$Q^a_{\alpha} Q^{\alpha}_b = \delta^a_b \tag{1.46}$$

$$A_{\alpha} Q^{\alpha}_a = 0 \ . \tag{1.47}$$

With these we can decompose any vector into a p-vector and a scalar:

$$k^a = Q^a_{\alpha} k^{\alpha} \tag{1.48}$$

$$k = A_{\alpha} k^{\alpha} \tag{1.49}$$

and conversely associate a vector with any p-vector:

$$m^{\alpha} = Q^{\alpha}_a m^a. \tag{1.50}$$

We now form the p-metric:

$$g_{ab} = Q^{\alpha}_a Q^{\beta}_b g_{\alpha\beta} \ . \tag{1.51}$$

The mixed Q-quantities project from five dimensions into four, orthogonal to A^{α}. The projection tensor is:

$$\begin{aligned} h_{\alpha}{}^{\beta} &= Q^a_{\alpha} Q^{\beta}_a \\ &= \delta^{\beta}_{\alpha} - A_{\alpha} A^{\beta}. \end{aligned} \tag{1.52}$$

Thus:

$$k^{\alpha} = Q^{\alpha}_a k^a + A^{\alpha} k \ . \tag{1.53}$$

We have several new types of derivative. First, there is the "A-derivative of a p-tensor":

$$k^{a\ldots}{}_{b\ldots,\gamma} A^{\gamma} \ . \tag{1.54}$$

If the A-derivative of $k^{a\ldots}{}_{b\ldots}$ vanishes it is "A-cylindrical" (the term will be more meaningful later). Then:

$$g_{mn,\alpha}A^{\alpha} = Q^{\rho}_{m}Q^{\sigma}_{n}(A_{\rho;\sigma} + A_{\sigma;\rho}) \tag{1.55}$$

$$F_{ab} = Q^{\rho}_{a}Q^{\sigma}_{b}(A_{\rho;\sigma} - A_{\sigma;\rho}) \tag{1.56}$$

$$B_{\rho} = A^{\sigma}(A_{\rho,\sigma} - A_{\sigma;\rho}) \tag{1.57}$$

$$F_{ab,\alpha}A^{\alpha} = Q^{\rho}_{a}Q^{\sigma}_{b}(B_{\rho,\sigma} - B_{\sigma,\rho}) \ . \tag{1.58}$$

There is the "p-derivative", a generalized derivative with respect to x^a:

$$k_{|a} = k_{,\alpha}Q^{\alpha}_{a} \ . \tag{1.59}$$

If k is A-cylindrical, then this is just the ordinary four-space coor - dinate derivative . In general p-derivatives do not commute:

$$\begin{aligned} &k_{|ab} - k_{|ba} = k_{,\sigma}(Q^{\sigma}_{a|b} - Q^{\sigma}_{b|a}) \\ &= k_{,\sigma}A^{\sigma}F_{ba}. \end{aligned} \tag{1.60}$$

There is the covariantderivative of a p-tensor:

$$k^{a}{}_{;\sigma} = k^{a}{}_{,\sigma} + \Gamma^{a}_{c\sigma}k^{c} \ . \tag{1.61}$$

Notice the mixed indices. In specifying $\Gamma^{b}_{a\sigma}$ it can be shown that the choice:

$$\Gamma^{b}_{a\sigma} = Q^{b}_{\beta}Q^{\alpha}_{a}\Gamma^{\beta}_{\alpha\sigma}(1c) - Q^{b}_{\sigma|a} \tag{1.62}$$

satisfies both of:

$$g_{ab;\alpha} = 0 \tag{1.63}$$

$$Q^{a}_{\alpha}Q^{\alpha}_{b;\sigma} = 0. \tag{1.64}$$

Meanwhile, the "covariant p-derivative" is the ordinary covariant derivative multiplied by Q_a:

$$k^{\alpha}{}_{;b} = k^{\alpha}{}_{|b} + \Gamma^{\alpha}_{\sigma b}k^{\sigma} \tag{1.65}$$

$$\Gamma^{\alpha}_{\sigma b} = \Gamma^{\alpha}_{\sigma\beta} Q^{\beta}_{b} \tag{1.66}$$

$$k^{a}{}_{;b} = k^{a}{}_{|b} + \Gamma^{a}_{cb}\, k^{c} \tag{1.67}$$

$$\Gamma^{a}_{cb} = \Gamma^{a}_{c\beta}\, Q^{\beta}_{b}$$

$$= \tfrac{1}{2} g^{ad}(g_{cd|b} + g_{db|c} - g_{cb|d}) \; . \tag{1.68}$$

Because there are several derivatives, there are several curvatures:

$$k^{\alpha}{}_{;\beta\gamma} - k^{\alpha}{}_{;\gamma\beta} = R^{\alpha}{}_{\delta\gamma\beta} k^{\delta} \tag{1.69}$$

$$k^{a}{}_{;\beta\gamma} - k^{a}{}_{;\gamma\beta} = R^{a}{}_{d\gamma\beta} k^{d} \tag{1.70}$$

$$R^{a}{}_{d\gamma\beta} = Q^{a}_{\alpha}\, Q^{\delta}_{d}\, (R^{\alpha}{}_{\delta\gamma\beta} + A^{\alpha}{}_{;\gamma} A_{\delta;\beta} - A^{\alpha}{}_{;\beta} A_{\delta;\gamma}) \tag{1.71}$$

$$k^{\alpha}{}_{;bc} - k^{\alpha}{}_{;cb} = Q^{\beta}_{b} Q^{\gamma}_{c} \left[R^{\alpha}{}_{\delta\gamma\beta} k^{\delta} + (A_{\gamma,\beta} - A_{\beta,\gamma})\, A^{\sigma} k^{\alpha}{}_{;\sigma} \right] \tag{1.72}$$

$$k^{a}{}_{;bc} - k^{a}{}_{;cb} = R^{a}{}_{dcb} k^{d} + \; k^{a}{}_{,d}\, A^{d} F_{cb} \; . \tag{1.73}$$

$R^{a}{}_{bcd}$ and $R^{\alpha}{}_{\beta\gamma\delta}$ are the standard four and five dimensional Riemann tensors, expressible in the usual way in terms of the respective connections:

$$R^{a}{}_{dcb} = Q^{\beta}_{b} Q^{\gamma}_{c} Q^{\delta}_{d} Q^{a}_{\alpha}\, (R^{\alpha}{}_{\delta\gamma\beta} + A^{\alpha}{}_{,\delta}(A_{\gamma,\beta} - A_{\beta,\gamma})$$

$$+ A^{\alpha}{}_{;\gamma} A_{\delta;\beta} - A^{\alpha}{}_{;\beta} A_{\delta;\gamma}) \; . \tag{1.74}$$

For Kaluza's theory we will use the double contraction of this:

$$\delta^{a}_{d} g^{bc} R^{d}{}_{cba} = R - g^{\gamma\sigma} A^{\tau}{}_{;\gamma} A_{\tau;\sigma} - (A^{\sigma}{}_{;\sigma})^{2}$$

$$+ B_{\sigma} B^{\sigma} - 2A^{\alpha}(A^{\sigma}{}_{;\sigma})_{,\alpha} + 2B^{\sigma}{}_{;\sigma} \; . \tag{1.75}$$

R is the five dimensional curvature scalar here.

Most of the theories single out "special" coordinate systems: those coordinate systems where the fifth dimension is not in evidence. These are identified by the demands that:

$$y^{a} = x^{a} \tag{1.76}$$

$$A^{\alpha} = (0,0,0,0,1). \tag{1.77}$$

Transformations between special coordinates are of the form:

$$\tilde{y}^a = f^a(y^b) \tag{1.78}$$

$$\tilde{y}^5 = y^5 + f^5(y^b) \tag{1.79}$$

coupled with:

$$\tilde{x}^a = f^a(x^b) \quad . \tag{1.80}$$

In special coordinates we have;

$$A_{\alpha} = (\phi_a \ , 1) \tag{1.81}$$

$$g_{\alpha\beta} = \begin{Bmatrix} g_{ab} + \phi_a\phi_b & ,\phi_a \\ \phi_b & , 1 \end{Bmatrix} \tag{1.82}$$

$$g^{\alpha\beta} = \begin{Bmatrix} g^{ab} & , -g^{ac}\phi_c \\ -g^{bc}\phi_c & , 1+g^{cd}\phi_c\phi_d \end{Bmatrix} \quad . \tag{1.83}$$

ϕ_a transforms according to:

$$\tilde{\phi}_a = \frac{\partial y^b}{\partial \tilde{y}^a} (\phi_b - f^5{}_{,b}) \tag{1.84}$$

so that f^5 induces a "gauge" transformation on ϕ_a (but not in the same sense as Weyl's).

In limiting our attention to special coordinates, the multi-faceted "four dimensions in five" structure we have just discussed becomes considerably restricted. "A-differentiation" becomes differentiation with respect to y^5:

$$k^{a\ldots}{}_{b\ldots;\alpha} A^{\alpha} = k^{a\ldots}{}_{b\ldots,5} \quad . \tag{1.85}$$

For p-differentiation:

$$Q^a_{\alpha} = \delta^a_{\alpha} \tag{1.86}$$

$$\left.\begin{aligned} Q^{\alpha}_a &= \delta^{\alpha}_a \quad ,\alpha = 1,2,3,4 \\ &= -\phi_a \quad ,\alpha = 5 \end{aligned}\right\} \tag{1.87}$$

$$k_{|a} = k_{,a} - \phi_a k_{,5} \quad . \tag{1.88}$$

Then:

$$F_{ab} = \phi_{a,b} - \phi_{b,a} + \phi_a \phi_{b,5} - \phi_b \phi_{a,5} \tag{1.89}$$

$$B_\alpha = (\phi_{\alpha,5}, 0) \tag{1.90}$$

$$A^\alpha{}_{;\alpha} = \Gamma^\alpha{}_{5\alpha} = \tfrac{1}{2}(\ln (\det g_{\alpha\beta}))_{,5} \tag{1.91}$$

and in fact:

$$\det g_{\alpha\beta} = \det g_{ab}, \tag{1.92}$$

$$\begin{aligned} A^\alpha{}_{;\alpha} &= \tfrac{1}{2}(\ln (\det g_{ab}))_{,5} \\ &= \tfrac{1}{2} g^{ab} g_{ab,5} \quad . \end{aligned} \tag{1.93}$$

Now, Kaluza's Theory assumes that the five-dimensional metric is A-cylindrical - a strong assumption. It implies that A_α is a Killing vector, B_α vanishes, and that the A-curves (the original x^a = constant curves) are geodesics. F_{ab} is A-cylindrical and satisfies:

$$F_{[ab,c]} = 0 \tag{1.94}$$

and, in special coordinates:

$$F_{ab} = \phi_{a,b} - \phi_{b,a} \quad . \tag{1.95}$$

To get field equations we assume a variational principle with the Lagrangian:

$$\mathcal{L} = \sqrt{|\det g_{\alpha\beta}|}\, R \tag{1.96}$$

in analogy with general relativity (gravitation only). Now:

$$R = \delta^a{}_d g^{bc} R^d{}_{cba} + \tfrac{1}{4}\phi_{ab}\phi^{ab} \quad . \tag{1.97}$$

In special coordinates we can replace det $g_{\alpha\beta}$ by det g_{ab}, and because the Lagrangian is A-cylindrical, the integral may be taken over a five-volume or over the four-volume domain of the parameters x^a :

$$\delta\int (g^{bc}R^a{}_{cba} + \tfrac{1}{4}\phi_{ab}\phi^{ab}) \sqrt{-g}\, d^4x = 0 \tag{1.98}$$

where:

$$g = \det g_{ab} \ . \tag{1.99}$$

The variation is subject to:

$$(\delta g^{ab})_{,5} = 0 \tag{1.100}$$

$$(\delta \phi_a)_{,5} = 0 \ . \tag{1.101}$$

This variational principle and the resulting field equations are identical to those of EMT.

The projective theories of Veblen and Hoffmann(13) and of Pauli(14), are reinterpretations of Kaluza's theory: these authors really only worked in different types of coordinates. The A-curves are assumed to represent only one point of the physical space. Then the metric must be A-cylindrical, and the field equations are again equivalent to EMT.

Since Kaluza's theory is equivalent to EMT, it does not predict anything new (and does not solve the problem of singularities, for example). We can of course expect new phenomena in modifying it. Einstein and Mayer(15) tried introducing a five dimensional formalism in four-space without actually introducing five-space. Then the "five-dimensional vectors" transform according to arbitary matrices M :

$$\tilde{V}^\alpha = M^\alpha{}_\beta V^\beta \ . \tag{1.102}$$

If these were true coordinates transformations $M^\alpha{}_\beta$ would be restricted by some differential identities. This construction creates a large number of new differential covariants, and it is difficult to attach meaning to all the new quantities.

Another interpretation of the formalism appears in theories in which the fifth dimension is closed. These were proposed by Einstein, Bergmann, and Bargmann starting in 1938(16-19). To constrict a space with closed fifth dimension we cut a thin slice out of a

five-dimensional continuum and identify the two open (four-dimensional) faces, resulting in a "tube". We assume that closed geodesics that go around the tube intersect themselves smoothly.

The term "thin" applied to the slice means: thin enough that there is exactly one closed geodesic in the fifth dimension through each point. The arc length, S, of one circuit of the closed geodesic, the "circumference" of the tube, is a constant of the space. The vectors tangent to these closed geodesics form the unit vector field A^{α} :

$$A^{\alpha}{}_{;\gamma} A^{\gamma} = 0 \ . \tag{1.103}$$

In special coordinates this becomes:

$$\phi_{a,5} = 0 \ . \tag{1.104}$$

The rest of the metric components are periodic in y^5, and indeed any field uniquely defined in the space is periodic in y^5 with period S. Then:

$$B_{\alpha} = 0 \tag{1.105}$$

$$F_{ab} = \phi_{a,b} - \phi_{b,a} \tag{1.106}$$

$$g_{ab,5} = A_{a;b} - A_{b;a} \ . \tag{1.107}$$

If we wish to derive field equations from a variational principle, we have four scalars of second differential order to choose from (all others differ from these by divergences) to construct the Lagrangian:

$$\delta^{a}{}_{d}\, g^{bc}\, R_{abc}{}^{d} \tag{1.108}$$

$$F^{ab} F_{ab} \tag{1.109}$$

$$(A_{\alpha;\beta} + A_{\beta;\alpha}) (A_{\gamma;\delta} + A_{\delta;\gamma}) g^{\alpha\beta} g^{\gamma\delta} \tag{1.110}$$

$$(A_{\alpha;\beta} + A_{\beta;\alpha}) (A_{\gamma;\delta} + A_{\delta;\gamma}) g^{\alpha\gamma} g^{\beta\delta} \ . \tag{1.111}$$

In the variation:

$$\delta\int W \sqrt{-g}\, d^5y = 0 \tag{1.112}$$

(this is stated in special coordinates, with $g = \det g_{ab}$) we must

preserve the geometry of the closed space. In special coordinates, ϕ_a must be independent of y^5, and g_{ab} must be periodic in y^5. These restrictions clash with the usual requirement that the variations vanish on the boundary of the domain of integration. Instead, we can ask that the boundary extend exactly once around the tube, and that the variations vanish on the part of the boundary generated by the A-curves. This modified procedure unfortunately results in integro-differential equations:

$$\int (P_{ab}\,\delta g^{ab} + J^a \delta\phi_a)\,\sqrt{-g}\,d^5y = 0 \tag{1.113}$$

which implies:

$$P_{ab} = 0 \tag{1.114}$$

$$\int_{y^5=0}^{s} J^a \sqrt{-g}\,dy^5 = 0 \;. \tag{1.115}$$

These satisfy five integro-differential identites:

$$\int_{y^5=0}^{s} (2\,P_a{}^b{}_{;b} + F_{ba}J^b)\sqrt{-g}\,dy^5 = 0 \tag{1.116}$$

$$\int_{y^5=0}^{s} (J^b{}_{;b} + P_{ab}g^{ab}{}_{,5})\,\sqrt{-g}\,dy^5 = 0 \;. \tag{1.117}$$

P_{ab} and J^a generally contain the expressions in the field equations of EMT (among other things).

The appearance of integro-differential equations can be objected to on the grounds that physical fields are usually described by pure differential equations. With a closed fifth dimension, such equations probably cannot be found in a straightforward way from a variational principle.

Einstein's Theory of The Nonsymmetric Field

The most popular unified field theory is Einstein's[20] (1945) "Theory of the Nonsymmetric Field". In it, one still has a four dimensional manifold, but the metric is no longer symmetric:

$$g_{ab} = g_{(ab)} + g_{[ab]} \tag{1.118}$$

$$g_{(ab)} = \tfrac{1}{2}\,(g_{ab} + g_{ba}) \tag{1.119}$$

$$g_{[ab]} = \tfrac{1}{2}\,(g_{ab} - g_{ba})\ . \tag{1.120}$$

The antisymmetric part is expected to be related to the electromagnetic field. The line element in the theory is expected to be:

$$\begin{aligned} ds^2 &= g_{ab}\ dx^a dx^b \\ &= g_{(ab)}\ dx^a dx^b\ . \end{aligned} \tag{1.121}$$

In the original proposal, Einstein raised the possibility that $g_{(ab)}$ is real while $g_{[ab]}$ is imaginary, so that g_{ab} then has "hermitian symmetry":

$$g_{ab} = \bar{g}_{ba} \tag{1.122}$$

where the bar indicates complex conjugation. The interpretation generally favoured, however, is that g_{ab} is strictly real and this is the interpretation we will use unless otherwise stated. Since the BMB theory is a modification of Einstein's, having different field equations in the same general framework, some discussion relevant to Einstein's theory will be found in later chapters. In chapter 3, we will discuss some of the interpretational problems of the nonsymmetric theories in general, including the particular problems of the hermitian symmetric alternative.

We assume a nonsymmetric connection :

$$\Gamma^a_{bc} = \Gamma^a_{(bc)} + \Gamma^a_{[bc]} \tag{1.123}$$

where $\Gamma^a_{(bc)}$ transforms as a connection, but $\Gamma^a_{[bc]}$ as a three index tensor, the "torsion". From the connection we find the curvature tensor:

$$R^a{}_{bcd} = \Gamma^a_{bc,d} - \Gamma^a_{bd,c} + \Gamma^a_{ed}\Gamma^e_{bc} - \Gamma^a_{ec}\Gamma^e_{bd} \tag{1.124}$$

and the contracted (on upper and last lower index) curvature tensor, no longer symmetric:

$$R_{ab} = R_{(ab)} + R_{[ab]}$$
$$= \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{ad}\Gamma^d_{cd} - \Gamma^c_{ab}\Gamma^d_{cb} \quad . \qquad (1.125)$$

Before we can identify the curvature scalar, we must specify the upper indexed metric. We have two choices:

$$g^{ac}\, g_{bc} = \delta^a_b \qquad (1.126)$$

$$g^{ac}\, g_{cb} = \delta^a_b \quad , \qquad (1.127)$$

both choices involving the assumption that $\det(g_{ab})$ is not zero, reasonable if the metric is to raise and lower indices. The former alternative is usually preferred (also it generates the proper EMT limit in BMB theory) and will be used in the rest of the thesis. Then:

$$R = g^{ab}\, R_{ab} \quad . \qquad (1.128)$$

For field equations, Einstein chose his Lagrangian in analogy with the pure gravitational field:

$$\delta \int B \sqrt{-g}\, d^4x = 0 \quad . \qquad (1.129)$$

We use B for the curvature and also L^a_{bc} for the connection in order to distinguish them from transformed quantities which will be used in the field equations. We do not assume any a priori relation between g_{ab} and L^a_{bc}, but vary them independently. The interdependence of g_{ab} and L^a_{bc} is one of the field equations. Performing the variation, we find:

$$g_{ab,c} - g_{ad}\, L^d_{cb} - g_{db}\, L^d_{ac} - \frac{2}{3}\, g_{ac} L^d_{[bd]} - \frac{2}{3}\, g_{ab} L^d_{[cd]} = 0 \quad . \qquad (1.130)$$

$$B_{ab}(L^c_{de}) = 0 \quad . \qquad (1.131)$$

In chapter 2, we will go through a derivation of the BMB field equations, from a slightly different Lagrangian . The field equations here can be found by work analogous to that in chapter 2[21].

This is an underdetermined set of field equations (coordinate transformation variations produce four identites), and further, the relation between g_{ab} and L^a_{bc} is more complicated than the one in

general relativity , which is:

$$g_{ab;c} = g_{ab,c} - g_{ad}\Gamma^{d}_{bc} - g_{db}\Gamma^{d}_{ac} = 0 \quad . \tag{1.132}$$

To get a usable set of equations we perform a "projective transformation":

$$\Gamma^{a}_{bc} = L^{a}_{bc} + \delta^{a}_{b} V_{c} \tag{1.133}$$

where V_c is any vector. Such a transformation preserves geodesics and parallel transported directions globally(22). However, the lengths of vectors undergoing parallel transport are changed. We choose:

$$V_{c} = \frac{2}{3} L^{d}_{[cd]} \tag{1.134}$$

(specifically to set $\Gamma^{d}_{[cd]}$ equal to zero). With this, our field equations become:

$$g_{ab,c} - g_{ad}\Gamma^{d}_{cb} - g_{db}\Gamma^{d}_{ac} = 0 \tag{1.135}$$

$$\Gamma^{c}_{[ac]} = 0 \tag{1.136}$$

$$R_{(ab)} = 0 \tag{1.137}$$

$$R_{[ab]} = V_{a,b} - V_{b,a} \tag{1.138}$$

The underdetermination of the system rests in eq. (1.138), in which V_a is not fully determined. Because of this, (1.138) is usually replaced by the equivalent statement:

$$R_{[[ab],c]} = 0. \tag{1.139}$$

With this system we can write R_{ab} as:

$$R_{ab} = \Gamma^{c}_{ab,c} - \frac{1}{2}\left(\Gamma^{c}_{(ac),b} + \Gamma^{c}_{(bc),a}\right) - \Gamma^{c}_{ad}\Gamma^{d}_{cb} + \Gamma^{c}_{ab}\Gamma^{d}_{cd} \tag{1.140}$$

to explicitly display the symmetry in a,b that $\Gamma^{a}_{(ac),b}$ now has. We note that (1.134) is equivalent to :

$$\left(\sqrt{-g}\, g^{[ab]}\right)_{,b} = 0 \quad . \tag{1.141}$$

Eq. (1.135) is not quite the same as (1.132). The indices in the second

term connection have been reversed, so that in fact:

$$g_{ab;c} = 2\, g_{ad} \Gamma^{d}_{[cb]} \ . \tag{1.142}$$

If we want the covariant dervative of g_{ab} to be zero we must set $\Gamma^{d}_{[cb]} = 0$. Then the connection is symmetric, there is always one symmetric g_{ab} which solves (1.135) (now equivalent to (1.132)), and we find the field equations are those of general relativity without electromagnetic field. It appears that $g_{ab;c}$ is necessarily nonzero in nonsymmetric unified field theory, although this will lead to difficulties in giving the metric a full geometric interpretation (see chapter 3).

We have not yet identified the representative of the electromagnetic field in this theory. We can expect some modification of Maxwell's equations, so we must anticipate a generalized electromagnetic field which approximates EMT's in appropriate circumstances. Simply on the basis of numbers of components we could say that $g_{[ab]}$ is either the electromagnetic field, or, on the same grounds, its dual. The field equations are not much help, unfortunately: $g_{[ab]}$ has the divergence equation (1.141) but $R_{[ab]}$ is the object with a potential, V_a.

In the absence of immediate identification of the electromagnetic field, we can look for it by examining the properties of the field equations. In particular, we can look for solutions representing single isolated charged particles. In the simplest case we ask for a spherically symmetric point mass geometry, asymptotically Minkowskian:

$$\lim_{r\to\infty} g_{(ab)} = \eta_{ab} \ . \tag{1.143}$$

For the electromagnetic field, we seek some tensor quantity related to $g_{[ab]}$ which goes nicely to zero as r tends to infinity. Let us not be as presumptuous as to ask for a regular solution yet. This type of solution would be comparable to the Reissner-Nordstrom geometry. Unfortunately

neither the use of $g_{[ab]}$ nor its dual as the electromagnetic field yields any such easily interpretable solution(23-26). This fact led Wyman(24) to suggest that the line element might not be so directly related to g_{ab} as we have assumed. This, however, is an intuitively unappealing thought.

Another approach to the electromagnetic field is to take the weak field limit (ie, linearization). Since EMT is valid for laboratory experiments on earth (a weak field situation), we expect to find the EMT equations in this limit. This in not the case: the linearized equations governing $g_{[ab]}$ are weaker than Maxwell's equations(25). Further, Papapetrou(23) showed that in second order the modification of $R_{(ab)}$ due to $g_{[ab]}$ is not in any way similar to that of the stress-energy-momentum tensor.

Yet another way to identify the electromagnetic field is by studying the equations of motion of isolated particles. This, however, is a difficult problem in itself. Being a large and complicated subject, we will not delve into the mathematical details here. In general relativity, the one body problem is nontrivial, and the two body problem is tractable only in approximation. Collections of particles in numbers large enough to be approximated as fluids are treated with the stress-energy-momentum tensor, T_{ab}, in Einstein's equations. Individual particles in a many-body problem are viewed as localities with nonzero (mass-distribution-type) T_{ab}, with vacuum (T_{ab} not necessarily zero) separating the individuals. Unified field theories prefer not to have a stress-energy-momentum tensor, so equation of motion work is usually done with isolated singularities. Most researchers rationalize that there should exist regular solutions, with form as yet unknown, asymptotically equivalent to singular solutions (all calculations

are done in far fields). Considering the scarcity of known completely regular vacuum solutions, the author believes that this attitude is hardly justified. It is possible to perform this kind of analysis in general relativity[27], but apparently the state of the art of equations of motion was not sufficiently advanced when Weyl's and five-dimensional theories were being actively studied, as very little such work was done in them. Early work in the nonsymmetric theory[28] suggested that it did not have the expected[29] equations of motion, but recent work[30] has been more encouraging. Johnson, in a series of papers[31], develops a very general approximation applicable to the study of equations of motion. From the application of his approximation to Einstein's theory he identifies the electromagnetic field as the dual of g^{ab}:

$$F_{ab} = \tfrac{1}{2} e_{abcd}\, g^{[cd]} \tag{1.143}$$

where e_{abcd} is Levi-Civita's tensor density. He states that this is the simplest expression approximating the field which governs the motion of charged particles at intermediate (laboratory) distances, whereas the true field governing motion everywhere is of complicated form[32]. With this identification, simple charged "particles" interact _via_ the Lorentz equation of motion _with_ radiation reaction over intermediate distances. The interaction at microscopic distances is complicated and not yet resolved; the interaction at astronomical distances involves an asymptotic field that does not vanish at infinity.

This appears to be the state of the standard nonsymmetric theory today. Work is still being done, particularly by Johnson, who is still applying his approximation scheme to the theory. One variant of the nonsymmetric theory, other than the BMB version, is Kursonoglu's[33]. We will just summarize his theory. He represents[34]:

$$g_{ab} = g_{(ab)} + \frac{1}{q}\, g_{[ab]} \tag{1.144}$$

where $g_{(ab)}$ is interpreted as the metric tensor, $g_{[ab]}$ as the generalized electromagnetic field, and q is a constant with the dimensions of an electric field. From a variational principle he derives field equations which can be put into the form:

$$R_{(ab)} = \tfrac{1}{2}K^2\left[\sqrt{\frac{-\det g_{(ab)}}{-\det g_{ab}}}\,(g_{(ab)} - \tau^{(cd)}g_{[ac]}g_{[bd]}) - g_{[ab]}\right] \tag{1.145}$$

$$R_{[[ab],c]} + \tfrac{1}{2}K^2\, g_{[[ab],c]} = 0 \tag{1.146}$$

$$\underline{g}^{[ab]}{}_{,b} = 0 \tag{1.147}$$

$$g_{ab,c} - g_{ad}\Gamma^d_{cb} - g_{db}\Gamma^d_{ac} = 0 \tag{1.148}$$

where K is related to q by:

$$\frac{K^2}{q^2} = \frac{4G}{c^4} \quad . \tag{1.149}$$

Here g^{ab} is the inverse of $g_{ab,}$ and $g^{[ab]}$ is the antisymmetric part, but $\tau^{(ab)}$ is the inverse of $g_{(ab)}$. Eq.(1.147) is again equivalent to demanding that torsion be traceless. Kursonoglu claims to have correct equations of motion and a class of completely regular solutions which have complicated magnetically charged structure.

Chapter II

Modification of the Nonsymmetric Theory

History

If we are discouraged by the behavior of Einstein's nonsymmetric theory (even today the best conceivable case implies that charged particles have a nonzero asymptotic electromagnetic field), we have the option of modifying it in the hope of finding better properties. In 1954, Bonnor[35] proposed modifying the Lagrangian in Einstein's variational principle to improve the equations of motion. His variational principle was equivalent to:

$$\delta\int\left(\sqrt{-g}\, g^{ab}R_{ab} + \frac{4\pi}{k^2}\sqrt{-g}\, g^{[ab]}g_{[ab]}\right) d^4x = 0 \tag{2.1}$$

where k is a universal constant with dimensions of length. With this it was found that the equations of motion became acceptable in a slow motion approximation scheme. He proposed the interpretation that the electromagnetic field is the dual of $g^{[ab]}$ (as C.R. Johnson does in the usual theory (eq (1.142))), but was unable to find a reasonable point charge solution.

Moffat and Boal[36] (1974) took the interpretation that $g_{[ab]}$ itself is proportional to the electromagnetic field and used the Lagrangian of eq. (2.1) to find the field equations (we will go through the derivation shortly):

$$g_{ab,c} - g_{ad}\Gamma^{d}_{cb} - g_{db}\Gamma^{d}_{ac} = 0 \tag{2.2}$$

$$\Gamma^{c}_{[ac]} = 0 \tag{2.3}$$

$$R_{(ab)} + I_{(ab)} = 0 \tag{2.4}$$

$$R_{[ab]} + I_{[ab]} = V_{a,b} - V_{b,a} \tag{2.5}$$

where:

$$I_{ab} = \frac{-4\pi}{k^2}\left(g_{ac}\, g^{[cd]}g_{db} + \tfrac{1}{2}\, g_{ab}\, g_{[cd]}g^{[cd]} + g_{[ab]}\right). \tag{2.6}$$

Compare this set to eq. (1.135)-(1.138), in the same notation: the modification is the introduction of I_{ab}. Again, we prefer:

$$_{[}(R_{[ab]} + I_{[ab]})_{,c]} = 0 \qquad (2.7)$$

to eq.(2.5), since the potential V_a is not determined by the field equations. Here eq.(2.3) is equivalent to:

$$(\sqrt{-g}\, g^{[ab]})_{,b} = 0 \ . \qquad (2.8)$$

Note that because the Lagrangian now has two terms, the second being electromagnetic and having a new constant k, this theory is not as nicely unified as Einstein's nonsymmetric theory. We hope that this will be offset by better behavior of the new theory.

Although Bonnor's treatment of the equations of motion also applies to this identification of F_{ab}, in a more modern analysis Johnson[37] showed that in his approximation scheme, charged particles interact in first nontrivial order according to Maxwell's equations and the Lorentz force law with radiation reaction, but without the asymptotically nonzero field found in Einstein's UFT.

Moffat and Boal found a static, spherically symmetric, electrically charged solution, which looks like a generalization of EMT's Reissner-Nordstrom solution[38]:

$$ds^2 = \left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)}$$

$$- r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (2.9)$$

where m is mass and Q is charge. The only nonzero element of $g_{[ab]}$ in the solution is:

$$g_{[14]} = \frac{kQ}{r^2} \ . \qquad (2.10)$$

On the basis of this we take the interpretation that $g_{[ab]}$ is the electromagnetic field multiplied by k:

$$g_{[ab]} = kF_{ab} \quad . \tag{2.11}$$

Then, if k = 0, this solution becomes exactly the Reissner-Nordstrom solution, suggesting the possibility that the field equations might become EMT's in the limit where k tends to zero. In fact this is the case if we take the limit of (2.2),(2.4),(2.5),(2.8):

$$g_{ab;c} = 0 \tag{2.12}$$

$$R_{ab} = 8\pi T_{ab} \tag{2.13}$$

$$\frac{-8\pi}{k} F_{ab} = V_{a,b} - V_{b,a} \tag{2.14}$$

$$(\sqrt{-g}\, F^{ab})_{,b} = 0 \tag{2.15}$$

$$T_{ab} = F_a{}^d F_{db} + \tfrac{1}{4} g_{ab} F_{cd} F^{cd} \quad . \tag{2.16}$$

Taking the limit is not quite a straightforward procedure. The small k limit of eq.(2.8) is a trivial equation, because there is an overall factor of k. In order to get the electromagnetic divergence equation we must ask for a nontrivial equation in the limit (one way is to say that k is very small but not zero). Nonetheless, this limit is an interesting property not found in Einstein's UFT. From eq.(2.14) we identify the "generalized electromagnetic potential" vector, A_a, by:

$$V_a = \frac{-8\pi}{k} A_a \quad , \tag{2.17}$$

although in the $k \neq 0$ case it is not directly related to $g_{[ab]}$, but rather to $R_{[ab]}$.

The line element stated in eq.(2.9) is in the real metric interpretation. We can do the same work with a hermitian symmetric metric obtaining the results that k becomes imaginary (ik) and the solution line element becomes:

$$ds^2 = \left(1 - \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)}$$

$$- r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \; . \tag{2.18}$$

Inside the radius kQ the signature of the metric changes from the usual (+---) to (----), which is a pure spacelike signature, so no particle following a timelike path can ever penetrate into this region, in particular to get to the central singularity. Moffat[39] has succeeded in showing that this solution is world-line complete, and since the singularity is not accessible to real particles, the physical space is completly regular. Particles which penetrate to radius kQ are not deflected back into the same space, but move onto a second Riemann sheet of the maximally extended geometry[40]. Having such a "regular" solution, the hermitian symmetric alternative may prove to have some advantage over the real metric theory. We will retain the real metric interpretation (unless otherwise stated) for reasons discussed in chapter 3.

For discussion of magnetic monopoles see "Duality and Magnetic Monopoles", later in this chapter. In other work done to date the first steps towards finding a charged spinning (analogue of the Kerr-Newman) solution have been discussed[41], and several extensions of the theory have been proposed[42].

Derivation of the Field Equations

Let us now derive the field equations from a variational principle[21]. We choose the Lagrangian in analogy with EMT as follows:

$$\mathcal{L} = \sqrt{-g}\, g^{ab} B_{ab} - \left(\frac{4\pi}{k^2}\right) \sqrt{-g}\, g^{[ab]} g_{[ab]} \quad , \tag{2.19}$$

where the contracted curvature is denoted by B_{ab}, and the connection by L^a_{bc} as in eq.(1.129)-(1.131). The quantities g^{ab} and L^a_{bc} are to be varied independently. We have written the formal variation several

times; taking the δ inside the integral:

$$\int \left[B_{ab}\, \delta(\underline{g}^{ab}) + \underline{g}^{ab}\, \delta B_{ab} - \frac{4\pi}{k^2}\, \delta(\underline{g}^{[ab]} g_{[ab]}) \right] d^4x = 0. \tag{2.20}$$

Let:

$$-\frac{4\pi}{k^2}\, \delta(\underline{g}^{[ab]} g_{[ab]}) = I_{ab}\, \delta \underline{g}^{ab}. \tag{2.21}$$

so that:

$$\int (B_{ab} + I_{ab})\, \delta \underline{g}^{ab} + \underline{g}^{ab}\, \delta B_{ab} \quad d^4x = 0. \tag{2.22}$$

Since B_{ab} depends only on the connection, its variation is independent of $\delta \underline{g}^{ab}$, while $\delta \underline{g}^{ab}$ is obviously independent of δL^a_{bc}:

$$\int (B_{ab} + I_{ab})\, \delta \underline{g}^{ab}\, d^4x = 0 \tag{2.23}$$

$$\int \underline{g}^{ab}\, \delta B_{ab}\, d^4x = 0 . \tag{2.24}$$

Eq. (2.23) implies:

$$B_{ab} + I_{ab} = 0. \tag{2.25}$$

Let us now determine I_{ab}.

$$\delta(-\underline{g}^{[ab]} g_{[ab]}) = g_{[ba]}\, \delta \underline{g}^{ab} + \underline{g}^{[ab]}\, \delta g_{ba} . \tag{2.26}$$

Now:

$$\delta g_{ad} = -g_{bd}\, g_{ac}\, \delta g^{bc} , \tag{2.27}$$

and from properties of determinants:

$$\delta g = -g\, g_{ab}\, \delta g^{ab} , \tag{2.28}$$

so:

$$\delta \underline{g}^{ab} = \sqrt{-g}\, \delta g^{db} + \frac{1}{2\sqrt{-g}}\, g^{ab}\, \delta g \tag{2.29}$$

$$\sqrt{-g}\, \delta g^{ab} = \delta \underline{g}^{ab} + \tfrac{1}{2}\sqrt{-g}\, g^{ab}\, g_{cd}\, \delta g^{cd} . \tag{2.30}$$

Then:

$$\begin{aligned} \delta(-\underline{g}^{[ab]} g_{[ab]}) &= g_{[ba]}\, \delta \underline{g}^{[ab]} - g_{ac}\, g_{db}\, g^{[cd]}\, \delta \underline{g}^{ab} \\ &-\tfrac{1}{2}\sqrt{-g}\, g_{dc} g^{[cd]} g_{ab}\, \delta g^{[ab]}. \end{aligned} \tag{2.31}$$

Now:

$$\sqrt{-g}\, g_{ab}\, \delta g^{ab} = -g_{ab}\, \delta(\sqrt{-g}\, g^{ab}) \tag{2.32}$$

implies:

$$\delta(-\underline{g}^{[ab]} g_{[ab]}) = g_{[ba]} \delta \underline{g}^{ab} - g_{ac} g^{[cd]} g_{db} \delta \underline{g}^{ab} + \tfrac{1}{2} g^{[cd]} g_{cd} g_{ab} \delta \underline{g}^{ab} \quad . \tag{2.33}$$

Therefore:

$$I_{ab} = \frac{-4\pi}{k^2}\left[g_{ac} g^{[cd]} g_{db} + \tfrac{1}{2} g_{ab} g_{[cd]} g^{[cd]} + g_{[ab]} \right] \quad . \tag{2.34}$$

For the variation of the curvature we use the variation of the covariant derivative of the connection:

$$(\delta L^a_{bc})_{;d} = \delta L^a_{bc,d} + L^a_{ed}\, \delta L^e_{bc} - L^e_{bd}\, \delta L^a_{ec} - L^e_{cd}\, \delta L^a_{be} \tag{2.35}$$

to find:

$$\begin{aligned} \delta B_{ab} &= -\delta L^c_{ab,c} + \delta L^c_{ac,b} + L^d_{ac}\, \delta L^c_{db} + L^c_{db}\, \delta L^d_{ac} \\ &\quad - L^c_{dc}\, \delta L^d_{ab} - L^d_{ab}\, \delta L^c_{dc} \\ &= -(\delta L^c_{ab})_{;c} + (\delta L^c_{ac})_{;b} + 2L^d_{[cb]} \delta L^c_{ad} \quad . \end{aligned} \tag{2.36}$$

Using:

$$L_a = L^c_{[ac]} \tag{2.37}$$

$$0 = \int \underline{g}^{ab} \left(-(\delta L^c_{ab})_{;c} + (\delta L^c_{ac})_{;b} + 2L^d_{[cb]} \delta L^c_{ad}\right) d^4x \tag{2.38}$$

we find, after some algebra:

$$0 = \int \left[\underline{g}^{ab}{}_{;c} - 2\, \underline{g}^{ab}\, L_c + 2\, \underline{g}^{ad}\, L^b_{[cd]} - \delta^b_c\, (\underline{g}^{ad}{}_{;d} - 2\underline{g}^{ad}\, L_d) \right] \delta L^c_{ab}\, d^4x \quad . \tag{2.39}$$

Define:

$$P^{ab}{}_c = \underline{g}^{ab}{}_{;c} - 2\underline{g}^{ab} L_c + \frac{2}{3} \delta^b_c \underline{g}^{ad} L_d + 2\underline{g}^{ad} L^b_{[cd]} \tag{2.40}$$

and (2.39) becomes:

$$\int (P^{ab}{}_c - \delta^b_c\, P^{ad}{}_d)\, \delta L^c_{ab}\, d^4x = 0 \quad . \tag{2.41}$$

Therefore $P^{ab}{}_c$ must vanish. With a little calculation we can find the equivalent statement in $g_{ab,c}$ rather than $\underline{g}^{ab}{}_{,c}$:

$$0 = g_{ab,c} - g_{ad}L^{d}_{cb} - g_{db}L^{d}_{ac} - \frac{2}{3}\, g_{ac}\, L^{d}_{[bd]} - \frac{2}{3}\, g_{ab}L^{d}_{[cd]} \tag{2.42}$$

which we have already seen as eq.(1.130).

Now, as discussed in chapter 1, we perform a projective transformation:

$$\Gamma^{a}_{bc} = L^{a}_{bc} + \frac{2}{3}\,\delta^{a}_{b}L_{c} \tag{2.43}$$

(which makes the new torsion, $\Gamma^{c}_{[ab]}$, tracelesss). It is fairly easy to see that:

$$R_{ab} = B_{ab} - \frac{2}{3}\,(L_{a,b} - L_{b,a}) \tag{2.44}$$

and that (2.40) and (2.42) respectively are equivalent to:

$$\underline{g}^{ab}{}_{,c} + \underline{g}^{db}\,\Gamma^{a}_{dc} + \underline{g}^{ad}\,\Gamma^{b}_{cd} - \tfrac{1}{2}\,\underline{g}^{ab}\,(\Gamma^{d}_{dc} + \Gamma^{d}_{cd}) = 0 \tag{2.45}$$

$$g_{ab,c} - g_{db}\,\Gamma^{d}_{ac} - g_{ad}\,\Gamma^{d}_{cb} = 0\ . \tag{2.46}$$

Gathering everything together our field equations are eq.(2.46) together with:

$$R_{(ab)} + I_{(ab)} = 0 \tag{2.47}$$

$$R_{[ab]} + I_{[ab]} = \frac{2}{3}\,(L_{a,b} - L_{b,a}) \tag{2.48}$$

$$\Gamma^{c}_{[ac]} = 0\ , \tag{2.49}$$

where I_{ab} is determined by (2.34). Now, L_{a} is not fully determined, so we merely identify a potential vector for $R_{[ab]}$:

$$V_{a} = \frac{2}{3}\,L_{a} \tag{2.50}$$

$$R_{[ab]} + I_{[ab]} = V_{a,b} - V_{b,a}\ . \tag{2.51}$$

This completes the derivation of the basic field equations, (2.7) following immediately from (2.51). To verify (2.8):

$$(\underline{g}^{[ac]})_{,c} = 0$$

we subtract the contraction of (2.45) on a and c from the contraction on b and c:

$$\begin{aligned} 0 &= 2\,\underline{g}^{[ac]}{}_{,c} + \tfrac{1}{2}\,\underline{g}^{ad}\,\Gamma^{c}_{cd} - \tfrac{1}{2}\,\underline{g}^{da}\,\Gamma^{c}_{dc} - \tfrac{1}{2}\,\underline{g}^{ad}\,\Gamma^{c}_{dc} + \tfrac{1}{2}\,\underline{g}^{da}\,\Gamma^{c}_{cd} \\ &= 2\,\underline{g}^{[ac]}{}_{,c} + 2\,\underline{g}^{(ad)}\,\Gamma^{c}_{cd} \end{aligned} \tag{2.52}$$

$$0 = \mathfrak{g}^{[ac]}{}_{,c} \; . \tag{2.53}$$

Note that covariant differentiation with repect to L^a_{bc} was used significantly in this derivation (in steps done to find eq. (2.39)). Covariant differentiation with respect to Γ^a_{bc} is not the same operation. Physical interpretation will require that we single out one covariant derivative, and the choice made may be hard to justify. Einstein and Straus [20] were able to derive their field equations (1.135)-(1.138) from a Lagrangian without a projective transformation, by including extra terms in the Lagrangian with Lagrange multipliers to obtain the traceless torsion and divergence equation on $\mathfrak{g}^{[ab]}$. With a similar derivation in BMB theory we would not need to consider L^a_{bc} . We will return to consideration of covariant differentiation in chapter 3.

Duality and Magnetic Monopoles

It is well known that Maxwell's vacuum equations are dual invariant, that is, if F_{ab} is a particular solution of them, then another solution is:

$$F'_{ab} = F_{ab} \cos\alpha + \overset{*}{F}_{ab} \sin\alpha \tag{2.54}$$

where α is an arbitrary angle, and $\overset{*}{F}_{ab}$ is the dual of F_{ab}:

$$\overset{*}{F}_{ab} = \tfrac{1}{2} \varepsilon_{abcd} F^{cd} \tag{2.55}$$

(ε_{abcd} is the Levi-Civita alternating symbol). Eq. (2.54) is called a "dual rotation" because it is reversible in the sense of a rotation:

$$F_{ab} = F'_{ab} \cos(-\alpha) + \overset{'*}{F}_{ab} \sin(-\alpha). \tag{2.56}$$

The key relation that makes eq(2.56) work is:

$$\begin{aligned} \overset{**}{F}_{ab} &= \tfrac{1}{2} \varepsilon_{abcd} F^{*cd} \\ &= \tfrac{1}{4} \varepsilon_{abcd} \varepsilon_{efhj} \eta^{ce} \eta^{df} \eta^{hk} \eta^{jl} F_{kl} \\ &= \delta^{l}_{[a} \delta^{k}_{b]} F_{kl} = -F_{ab} \; . \end{aligned} \tag{2.57}$$

The dual of the dual is the negative of the original field. If we relax the usual source equations to allow the possibility of magnetic charge, the complete Maxwell equations (with sources) become dual invariant, and such rotations (by $\alpha = \frac{\pi}{2}$.) generate magnetic monopoles out of electric monopoles. Dirac showed that a viable electromagnetic theory can be constructed with such monopoles, leading to charge quantization[43].

EMT's (vacuum) equations are similarly dual invariant, with:

$$\begin{aligned} F^*_{ab} &= \tfrac{1}{2}\, e_{abcd}\, F^{cd} \\ &= \tfrac{1}{2}\, e_{abcd}\, g^{ce}\, g^{df}\, F_{ef} \end{aligned} \tag{2.58}$$

where e_{abcd} is the alternating tensor:

$$e_{abcd} = \sqrt{-g}\, \varepsilon_{abcd} \tag{2.59}$$

$$\tfrac{1}{4}\, e_{abcd}\, e_{efhj}\, g^{ce}\, g^{df}\, g^{hk}\, g^{jl} = \delta^{l}_{[a}\, \delta^{k}_{b]} \tag{2.60}$$

$$e_{abcd;e} = 0 \ . \tag{2.61}$$

Therefore every electrically charged monopole solution has a corresponding Dirac monopole solution. The general electrically and magnetically charged Reissner-Nordstrom solution is then[44]:

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m}{r} + \frac{4\pi(Q^2 + L^2)}{r^2}\right) dt^2 \\ &- \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{4\pi(Q^2 + L^2)}{r^2}\right)} - r^2(\, d\theta^2 + \sin^2\theta\, d\phi^2\,) \end{aligned} \tag{2.62}$$

$$E_r = \frac{Q}{r^2} \quad , \quad B_r = \frac{L}{r^2} \ . \tag{2.63}$$

In proposing their theory Moffat and Boal[36] suggested that their equations, having an electric monopole solution, probably do not have an equivalent magnetic monopole. If this were indeed the case, it would be a favourable property of the theory, as no magnetic monopoles have yet

been detected in nature[43]. No such solution has been found in BMB theory by studying plausible trial solutions involving polynomial expansions in the radial coordinate[45].

Let us consider dual rotations in BMB theory. The lack of an obvious fundamental tensor for raising and lowering indices (see chapter 3), the inclusion of the electromagnetic field in the metric, and the lack of any covariantly constant tensor related to the metric make it difficult to define a usable duality rotation. In analogy with EMT we formally write:

$$g'_{[ab]} = g_{[ab]} \cos\alpha + \overset{*}{g}_{[ab]} \sin\alpha . \tag{2.64}$$

There are a number of fairly simple possible forms to choose from for the dual, $\overset{*}{g}_{[ab]}$:

$$ {}^{1}\overset{*}{g}_{[ab]} = \tfrac{1}{2}\, e_{abcd}\, g^{[cd]} \tag{2.65}$$

$$ {}^{2}\overset{*}{g}_{[ab]} = \tfrac{1}{2}\, e_{abcd}\, g^{ec}\, g^{df}\, g_{[ef]} \tag{2.66}$$

$$ {}^{3}\overset{*}{g}_{[ab]} = \tfrac{1}{2}\, e_{abcd}\, g^{ce}\, g^{df}\, g_{[ef]} \tag{2.67}$$

$$ {}^{4}\overset{*}{g}_{[ab]} = \tfrac{1}{2}\, e_{abcd}\, g^{(ce)}\, g^{(df)}\, g_{[ef]} \tag{2.68}$$

$$ {}^{5}\overset{*}{g}_{[ab]} = \tfrac{1}{2}\, e_{abcd}\, \tau^{ce}\, \tau^{df}\, g_{[ef]} , \tag{2.69}$$

where τ^{ab} is the symmetric tensor satisfying:

$$\tau^{ab}\, g_{(cb)} = \delta^{a}_{c} , \tag{2.70}$$

and:

$$ {}^{6}\overset{*}{g}_{[ab]} = \tfrac{1}{2}\, e_{abcd}\, \phi^{cd} , \tag{2.71}$$

where:

$$\phi^{ab}\, g_{[cb]} = \delta^{a}_{c} \tag{2.72}$$

(ϕ^{ab} is antisymmetric). If we decide g_{ab} is the fundamental geometric tensor(see chapter 3), then ${}^{1}\overset{*}{g}_{[ab]}$ is the most natural choice, but $g^{[cd]}$ does not formally look like $g_{[ab]}$ with two indices raised. Eq.(2.66) -

(2.69) seem closer formally to the EMT dual, ${}^5g^*_{[ab]}$ being the most likely if $g_{(ab)}$ is the fundamental tensor (and ${}^4g^*_{[ab]}$ possible in this case). Eq.(2.66) and (2.57) are alternates to (2.65). ${}^2g^*_{[ab]}$ has a more natural order of summation indices in view of definition of g^{ab}, but the "upper index field" is not antisymmetric. In ${}^3g^*_{[ab]}$, the upper index field is still skew. ${}^6g^*_{[ab]}$ might be chosen if nothing else is suitable.

Let us look in more detail at each of these proposed duals. Under scrutiny ${}^6g^*_{[ab]}$ can quickly be eliminated. The determinant of $g_{[ab]}$ can be zero (example: the electric monople exact solution), so ${}^6g^*_{[ab]}$ does not always exist. Further, when it does exist, we find:

$$\phi \equiv \det g_{[ab]} \tag{2.73}$$

$$\phi^{ab} = \frac{\sqrt{-g}}{2\sqrt{\phi}}\, e^{abcd}\, g_{[cd]} \tag{2.74}$$

$${}^6g^*_{[ab]} = \frac{\sqrt{-g}}{\sqrt{\phi}}\, g_{[ab]} \ . \tag{2.75}$$

This does not become the usual dual in the small k limit. To investigate ${}^1g^*_{[ab]}$, $g^{[ab]}$ can be written(46) :

$$g^{[ab]} = \frac{\phi}{-g}\,\phi^{ab} + \frac{\tau}{-g}\,\tau^{ac}\tau^{bd}\, g_{[cd]} \tag{2.76}$$

where:

$$\tau \equiv \det g_{(ab)} \tag{2.77}$$

when ϕ is nonzero (ϕ zero must be treated separately). With this:

$${}^1g^*_{[ab]} = \frac{\sqrt{\phi}}{\sqrt{-g}}\, g_{[ab]} + \frac{\tau}{-g}\,\left({}^5g^*_{[ab]}\right) , \tag{2.78}$$

giving a dually transformed field (replacing the prime with the number of the dual chosen):

$$\begin{aligned} {}^1g_{[ab]} &= g_{[ab]}\left(\cos\alpha + \frac{\sqrt{\phi}}{\sqrt{-g}}\sin\alpha\right) \\ &+ \frac{\tau}{-g}\left(\tfrac{1}{2}\, e_{abcd}\,\tau^{ce}\,\tau^{df}\, g_{[ef]}\right)\sin\alpha . \end{aligned} \tag{2.79}$$

This implies:

$${}^1\phi = \det\left({}^1g_{[ab]}\right) \tag{2.80}$$

$$\neq \phi$$

$$^1g = \det {}^1g_{ab} \tag{2.81}$$

$$\neq g \; .$$

The supposed inverse transformation is complicated:

$$^1({}^1g_{[ab]})^* = \frac{\sqrt{{}^1\phi}}{\sqrt{{}^1_{-}g}} \; ({}^1g_{[ab]}) + \frac{\tau}{{}^1_{-}g} \; \left(\tfrac{1}{2} \, e_{abcd} \, \tau^{ce} \tau^{df} \; ({}^1g_{[ef]}) \right) \tag{2.82}$$

$$Q_{[ab]} = {}^1g_{[ab]} \cos\alpha - {}^1({}^1g_{[ab]})^* \sin\alpha$$
$$= g_{[ab]} \left[\cos^2\alpha + \left(\frac{\sqrt{\phi}}{\sqrt{-g}} - \frac{\sqrt{{}^1\phi}}{\sqrt{{}^1_{-}g}} \right) \cos\alpha \sin\alpha - \frac{\sqrt{{}^1\phi\phi}}{\sqrt{{}^1gg}} \sin^2\alpha \right]$$
$$+ {}^5g^{*}_{[ab]} \left[-\tau \left(\frac{\sqrt{{}^1\phi}}{-g\sqrt{{}^1_{-}g}} + \frac{\sqrt{\phi}}{{}^1_{-}g\sqrt{-g}} \right) \sin^2\alpha + \tau \cos\alpha \sin\alpha \left(\frac{1}{-g} - \frac{1}{{}^1_{-}g} \right) \right] \tag{2.83}$$
$$- {}^5g^{**}_{[ab]} \frac{\tau^2}{{}^1gg} \sin^2\alpha$$

where:

$$^5g^{**}_{[ab]} = \tfrac{1}{4} \, e_{abcd} \, e_{efkl} \, \tau^{ce} \tau^{df} \tau^{kh} \tau^{lj} \, g_{[hj]} \; . \tag{2.84}$$

We have certainly not been returned to the original field and therefore $^1g^*_{[ab]}$ does not generate a dual rotation. (Since the ϕ nonzero general case fails, we will not worry about the special case of zero ϕ).

Note eq.(2.84): it is a "dual of the dual" expression analogous to eq.(2.56). This expression (in case(4)), or a similar expression , appears in the inverse transformation equations for all of cases (2) through (5). One of the conditions necessary to obtain a dual rotation is that the analogue of (2.60) (which implies (2.57)) hold, and this appears difficult to satisfy. In fact, there are other nontrivial problems in cases (2), (3) and (4), but this is the only obstacle in case (5). It is necessary that:

$$\varepsilon^{abcd}{}_{;e} \doteq 0 \tag{2.85}$$

(for example, this defines how tensor densities transform[47]), and the upper and lower alternating densities are related by their identities,

including:

$$\tfrac{1}{4}\varepsilon^{abcd}\,\varepsilon_{cdef} = \delta^{a}_{[f}\,\delta^{b}_{e]} \,. \tag{2.86}$$

Taking the covariant derivative of eq.(2.86) we find:

$$\varepsilon_{abcd;e} = 0 \,. \tag{2.87}$$

Now, because the covariant derivative of $g_{(ab)}$ (and hence τ^{ab}) is not zero, obviously:

$$\tfrac{1}{4}\, e_{abcd}\, e_{efkl}\, \tau^{ce}\tau^{df}\tau^{kh}\tau^{lj} \neq \delta^{a}_{[f}\,\delta^{b}_{e]} \tag{2.88}$$

and we conclude that none of the possibilities we have suggested generate dual rotations.

From the considerations of this section, it appears that there is no simple way to define a meaningful electromagnetic dual in nonsymmetric unified field theories, much less test dual invariance. While experiment presently indicates no symmetry between electric and magnetic charges, the acceptability of the total loss of the duality concept is a matter for investigation.

<u>Possible Special Case Solutions</u>

This section points out some special circumstances in BMB theory in which solutions may exist. These cases are special because they may shed more light on the differences between BMB and EMT, and possibly on the validity of BMB theory itself. Assumptions of a few simple particular forms has not led to any such solutions yet.

An important question to ask is: are there any solutions involving nonzero electromagnetic field:

$$g_{[ab]} \neq 0 \tag{2.89}$$

on a flat background? We will have to specify what we mean by "flat". We can ask for a Minkowskian $g_{(ab)}$:

$$g_{(ab)} = \eta_{ab} \tag{2.90}$$

or that there exist coordinates in which geodesics are straight lines:

$$\Gamma^{a}_{(bc)} = 0 \qquad (2.91)$$

(everywhere) or that there be no curvature:

$$R^{a}{}_{bcd} = 0 \ . \qquad (2.92)$$

These are non-equivalent possibilities, becoming equivalent in the small k limit. Eq, (2.92) is certainly the strongest condition. No such flat background solutions exist in EMT: nonzero Maxwell field implies nonzero stress-energy-momentum tensor, and hence nonzero Ricci curvature. The BMB equations become EMT's as k tends to zero, so we can immediately say that any solution with:

$$\lim_{k\to 0}\left(\frac{g_{[ab]}}{k}\right) \neq 0 \qquad (2.93)$$

violates all of (2.90) - (2.93) because of the small k limit.

This last statement raises a more general question: are there solutions which do not have EMT analogies? Such solutions would become purely gravitational as k tended to zero, having all electromagnetic quantities functions of k which go to zero in the limit. If there is the possibility of such solutions, then I_{ab} must be studied to see if there are any circumstances in which we can have (2.89) and:

$$I_{(ab)} = 0 \qquad (2.94)$$

$$I_{[[ab],c]} = 0 \ . \qquad (2.95)$$

Only in such cases could (2.92) hold.

Wyman and Zassenhaus(48) have found all solutions with zero curvature in Einstein's nonsymmetric theory, most of them physically non-trivial. The analogous set in BMB promises to be reduced by the constraints (2.94),(2.95).

Another (possibly empty) family of solutions without EMT analogue would be:

$$g_{[ab]} \neq 0 \tag{2.96}$$

$$V_a = \phi_{,a} \tag{2.97}$$

(ϕ is any scalar field), which means:

$$R_{[ab]} = -I_{[ab]} \quad . \tag{2.98}$$

The small k limit implies:

$$\lim_{k\to 0}\left(\frac{g_{[ab]}}{k}\right) = 0 \quad . \tag{2.99}$$

Since there does not seem to be any obvious interpretation of this family, we cannot declare its members desirable or undesirable without further investigation.

Chapter III

Physical Interpretation of Geometry

General Relativity's Geometric Structure[49]

General relativity's geometric formulation is aesthetically appealing because almost all of the geometric structure has physical interpretation. There is very little extraneous geometry, very little of classical physics left out (although quantization is still a stumbling block).

The geometric structure is a four dimensional manifold with metric. The four dimensions are space (three) and time (one) which are bound together in the manifold, spacetime. The metric, g_{ab}, is a smooth (except in special circumstances) tensorfield on the manifold used to obtain the "inner-product" of two vectors:

$$g_{ab}\, k^a\, l^b \tag{3.1}$$

and the "length", a special case of this:

$$|k|^2 = g_{ab}\, k^a\, k^b \ . \tag{3.2}$$

Now, as the inner product is usually formulated (based on its beginnings in ordinary vector analysis), it is independent of the order of its arguments, and length is always non-negative. Then we can use it to define the "angle" between two vectors. General relativity retains the order independent nature (this means g_{ab} is symmetric), but to model light propagation properly, the non-negative length requirement is relaxed. To have length of consistent sign the metric must have consistent signature: (++++) or (----), in the Sylvester canonical form[50]. Relativity distinguishes time from space by switching one of the four diagonal elements: we choose the convention (+ - - -). Then there are three types of vector: spacelike ($|k|^2$ negative), pointing "in three space"; null ($|k|^2$zero), pointing along paths followed by light in four-space; and timelike ($|k|^2$ positive), pointing along massive particle paths. We can still use the inner product to identify the angle between two

spacelike or two timelike vectors, for example, and establish other relations for other cases. Physically, we interpret timelike magnitude as the rate at which time passes (as already mentioned) for an observer on the curve to which the timelike vector is tangent.

On a manifold an affine connection, Γ^{a}_{bc}, defines quantities such as parallel transport and covariant derivatives. The covariant derivative of tensor $Q^{a\ldots}{}_{b\ldots}$:

$$Q^{a\ldots}{}_{b\ldots;c} \tag{3.3}$$

is itself a tensor. The ordinary derivative:

$$Q^{a\ldots}{}_{b\ldots,c} = \frac{\partial}{\partial x^{c}}\left(Q^{a\ldots}{}_{b\ldots}\right) \tag{3.4}$$

is not. The form of the covariant derivative of a vector, for example, is:

$$k^{a}{}_{;b} = k^{a}{}_{,b} + k^{c}\Gamma^{a}_{cb} \quad . \tag{3.5}$$

Note the order of indices in the second term: the connection is not a priori symmetric. To parallel transport a tensor $Q^{a\ldots}{}_{b\ldots}$ along a curve with tangent k^{a}, the tensor is changed as it moves so that its covariant derivative in direction of k^{a} is zero:

$$Q^{a\ldots}{}_{b\ldots;c}\, k^{c} = 0 \ . \tag{3.6}$$

Mathematically, parallel transport is thought of as the way to move a tensor taking into account changes in local curvature without changing what the tensor actually represents. If we ask for those curves along which the tangent is itself parallel transported:

$$t^{a}{}_{;c}\, t^{c} = 0 \tag{3.7}$$

we find geodesics, the generalizations of straight lines on curved manifolds.

An important theorem is: if there is a smooth symmetric two-index tensorfield on a manifold, in particular, the metric, g_{ab}, then there is

a symmetric connection on the manifold uniquely determined by the condition that g_{ab} be preserved in parallel transport, ie:

$$\begin{aligned} 0 &= g_{ab;c} \\ &= g_{ab,c} - g_{ad}\Gamma^{d}_{bc} - g_{db}\Gamma^{d}_{ac} \end{aligned} \qquad (3.8)$$

(an equation we have already seen). It is called the Levi-Civita connection, and has the form:

$$\Gamma^{a}_{bc} = \tfrac{1}{2}\, g^{ad}\,(g_{dc,b} + g_{bd,c} - g_{bc,d}) \quad . \qquad (3.9)$$

With this connection inner products (and hence, lengths) are preserved in parallel propagation. Inner product preservation is consistent with the intuitive idea that parallel transport moves vectors maintaining their intrinsic nature. Further, it is easy to show that the curves that extremize arc length between pairs of points, that is, which are mathematically the solutions of[51]:

$$\delta \int_{a}^{b} ds = 0 \qquad (3.10)$$

$$ds^2 = g_{ab}\, dx^a dx^b \quad , \qquad (3.11)$$

are geodesics of the Levi-Civita connection. This reinforces the concept that geodesics are the generalizations of straight lines.

The connection defines the curvature tensor:

$$R^{a}{}_{bcd} = \Gamma^{a}_{bd,c} - \Gamma^{a}_{bc,d} + \Gamma^{a}_{ec}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{bc} \qquad (3.12)$$

and its contractions:

$$R_{ab} = R^{c}{}_{acb} \qquad (3.13)$$

$$R = g^{ab} R_{ab} \quad . \qquad (3.14)$$

The curvature satisfies several identities, in general relativity:

$$R^{a}{}_{bcd} = R^{a}{}_{b[cd]} \qquad (3.15)$$

$$R^{a}{}_{[bcd]} = 0 \qquad (3.16)$$

$$R^{a}{}_{b[cd;e]} = 0 \qquad (3.17)$$

(and others when a is lowered). (3.17) is the so called Bianchi identity. Note that $R^{a}{}_{bcd}$ and R_{ab} are independent of g_{ab} until we relate g_{ab} and

Γ^{a}_{bc} using (3.8). $R^{a}_{\ bcd}$ measures local gravitational effects through the geodesic deviation equation. For a geodesic vectorfield, k^a, in some locality, representing a group of freely falling test particles, the distance between nearby particles is described by a vectorfield p^a which is formed so that the Lie derivative(52) of p^a along integral curves of k^a is zero:

$$\mathcal{L}_{(k)}\, p^b = 0 \quad . \tag{3.18}$$

Then, the relative acceleration of two nearby test particles is:

$$k^b\, k^c\, p^a_{\ ;bc} = R^a_{\ bcd}\, k^b\, k^c\, p^d \ . \tag{3.19}$$

This is the geodesic deviation equation in general relativity.

In the investigation of mechanics, we regularly use the concept of the "frame". An observer has a space coordinate system set up around himself and measures events with respect to a clock carried with him. This space coordinate system and this time constitute the observer's frame. Those frames in which local space appears simplest (fictitious forces are eliminated) are called the freely falling frames. Geometrically a freely falling frame is postulated to follow a timelike geodesic (in the idealization that the observer's laboratory is of point size). The are length between two points on the geodesic is proportional to the elapsed proper time. The observer's three coordinate basis vectors, which do not vary as viewed by the observer, are three mutually orthogonal space-like vectors all orthogonal to the tamgent, which are parallel propagated along the geodesic. At any spacetime event it is possible to set up a frame in which the (symmetric) connection is zero at the point. Further, in general relativity one can show that the metric has the flat Minkowski form there:

$$g_{ab} = \eta_{ab} \quad . \tag{3.20}$$

even though the curvature may be nonzero. From this result we say that the physical space is always locally flat.

Symmetries of a particular solution are identified by Killing vectorfields. If a solution has some continuous symmetry (eg.,it does not evolve in time (is stationary)or has rotational symmetries), then there is a vectorfield, k^a, such that an observer moving along its integral curves does not see the geometry change. Mathematically:

$$\mathcal{L}_{(k)}\, g_{ab} = 0 \ . \tag{3.21}$$

This can easily be rewritten as the usual form of Killing's equation in general relativity:

$$k_{(a;b)} = 0 \ . \tag{3.22}$$

Killing vectorfields generate conserved quantities in geodesic motion. The inner product of the tangent, t^a, of any geodesic with any Killing vectorfield, k^a, that exists in a particular solution, is constant along the geodesic:

$$t^c(g_{ab}\, k^a\, t^b)_{;c} = 0 \ . \tag{3.23}$$

If k^a is timelike (a stationary solution), the inner product is related to the total energy of the particle in the field; if the solution is axially symmetric , the angular momentum of the particle is related to the inner product of the tangent and a killing vector.

Parallel Transport and a Real Metric

Let us begin our study of the geometry of BMB theory with consideration of the affine connection and parallel transport. The first problem to face is the existence of several connections, only one of which (at most) can represent the parallel transport which produces physical effects. There is L^a_{bc} , Γ^a_{bc} (the metric is not compatible with either of these) and there is the Levi-Civita connection, now labeled $\Gamma^a_{bc(lc)}$,

compatible with $g_{(ab)}$. Rewriting eq.(3.9):

$$\Gamma^a_{bc(lc)} = \tfrac{1}{2}\tau^{ad}\left(g_{(dc),b} + g_{(bd),c} - g_{(bc),d}\right) \tag{3.24}$$

with τ^{ad} defined by eq.(2.68). This connection is unrelated to L^a_{bc} or Γ^a_{bc} (54), and would need to be given serious consideration if $g_{(ab)}$ were the fundamental geometric tensor. However, since it seems artificial to separate $g_{(ab)}$ and $g_{[ab]}$ for geometric purposes, $\Gamma^a_{bc(lc)}$ is not likely the "physical" connection. As mentioned in discussing Einstein's nonsymmetric theory, L^a_{bc} and Γ^a_{bc} are related by a projective transformation given by eq.(2.43):

$$\Gamma^a_{bc} = L^a_{bc} + \tfrac{2}{3}\delta^a_b L_c \ .$$

These two connections therefore have the same geodesic paths (differently parametrized), and the same parallel propagated vector directions, but the lengths of parallel propagated vectors are different. Since the simplest set of field equations is in terms of Γ^a_{bc}, and since L^a_{bc} does not appear absolutely necessary to derive that set, Γ^a_{bc} seems to be the most likely candidate for the physical connection. These arguments are not all that strong, so the final judgement awaits work on equations of motion(55), light propagation (see chapter 5), or other physical properties.

It is difficult to give the metric tensor full geometric interpretation, primarily because it is not compatible with the connection:

$$g_{ab;c} = 2g_{ad}\Gamma^d_{[cb]} \ . \tag{3.25}$$

We begin with the use of a real g_{ab} (and will come to hermitian symmetry later) and immediately must decide whether g_{ab} or $g_{(ab)}$ is the fundamental geometric tensor. With $g_{(ab)}$ it is easy to define the general inner product, but such separation of $g_{(ab)}$ and $g_{[ab]}$ is contrary to unification, and we should really try to work with g_{ab}. Having made this choice we

find ambiguity in the inner product: the expected expression is not symmetric in its arguments:

$$g_{ab}\, k^a\, l^a \neq g_{ab}\, l^a\, k^b \quad . \tag{3.26}$$

For two arbitrary vectors, there is no way to single out one of the two possibilities. The ambiguity only disappears for magnitudes:

$$g_{ab}\, k^a\, k^b = g_{(ab)}\, k^a\, k^b \; . \tag{3.27}$$

It is not possible for the general inner product to be preserved in parallel transport (nor would it be if we defined it using $g_{(ab)}$, unless we also used $\Gamma^a_{bc(lc)}$, ignoring Γ^a_{bc}). If we parallel propagate vectors l^a, m^a along a vector k^a, the inner product changes according to:

$$k^c(g_{ab}\, l^a m^b)_{;c} = 2g_{ad}\Gamma^d_{[cb]} l^a m^b k^c \quad . \tag{3.28}$$

In general this is not zero, but it will be zero in the special case of m^a being k^a. This means that the length of the tangent to a geodesic is preserved:

$$k^c\, k^a_{\;;c} = 0 \tag{3.29}$$

$$k^c(g_{ab} k^a k^b)_{;c} = 0 \quad , \tag{3.30}$$

and one of the two possible inner products of a vector, l^a, parallel transported along a geodesic with the tangent, k^a, is preserved:

$$k^c(g_{ab}\, l^a k^b)_{;c} = 0 \tag{3.31}$$

$$k^c(g_{ab}\, k^a l^b)_{;c} \neq 0 \; . \tag{3.32}$$

These two conserved quantities can be used to simplify the calculation of particular geodesics and associated vectors in a solution. Note that the magnitude of a parallel propagated non-tangent is not usually preserved. This leads to an unexpected property: under appropriate circumstances a spacelike vector parallel propagated becomes null (at a single point), and then timelike[56]. To set up a frame, therefore, we cannot simply parallel propagate initially chosen spacelike vectors. Unfortunately there

is no alternate to parallel transport immediately obvious, so we do not presently know how to construct a frame's basis vectors.

There is another problem in trying to construct frames: we do not seem to have local flatness. Since we do not expect to be able to eliminate the electromagnetic field by changing coordinates (if $g_{[ab]}$ is nonzero in one set of coordinates, it is nonzero in all of them), we can only ask for Minkowskian $g_{(ab)}$ at a point:

$$g_{ab} = \eta_{ab} + g_{[ab]} \quad . \tag{3.33}$$

Then, however, eq.(2.2) is not solved by zero Γ^a_{bc} . In fact, $\Gamma^a_{[bc]}$ is a tensor, so it cannot be reduced to zero by coordinate transformations. If we look at coordinate transformations on $\Gamma^a_{(bc)}$, probably the strongest statement that can be made is that by appropriate choice of coordinates one can make the symmetric part of the connection and all totally symmetrized derivatives vanish at a single point[57]:

$$\Gamma^a_{(bc)} = 0 \tag{3.34}$$

$$\Gamma^a_{(bc,d)} = 0 \tag{3.35}$$

$$\Gamma^a_{(bc,d\ldots e)} = 0 \quad . \tag{3.36}$$

In eq.(2.2), however, these do not imply that $g_{(ab)}$ has constant coefficients. In addition to needing a method to construct well behaved frames we must decide whether it is physically reasonable that nonzero electromagnetic field means space is not locally flat, even as seen by a neutral test particle. Certainly space is locally flat for a neutral test particle in EMT.

We have already called the autoparallel curves geodesics in using eq. (3.29). It is more natural to call the generalized "straight line" the curve whose tangent is always parallel to itself, and in this case abandon extremization of arc length. Extremals of arc length still exist,

solutions of:

$$\delta \int ds = 0 \tag{3.37}$$

$$ds^2 = g_{ab}\, dx^a\, dx^b \ . \tag{3.38}$$

It is easy to show that these define the Levi-Civita symmetric connection on the manifold, with respect to which these curves are geodesics. Thus the Levi-Civita connection exists along with Γ^a_{bc} . We have already argued preference for Γ^a_{bc}, so to use extremals of arc length as free fall paths would be contradictory.

The curvatures $R^a{}_{bcd}$, R_{ab}, and R are still given by eq.(3.12)-(3.24), but R_{ab}, symmetric in EMT, now has both symmetric and antisymmetric parts. $R^a{}_{bcd}$ is still antisymmetric in c and d but the identities (3.16),(3.17) become[58]:

$$R^a{}_{[bcd]} = 4\, \underset{[\qquad\quad]}{\Gamma^e{}_{[dc]}\, \Gamma^a{}_{[be]}} + 2\, \Gamma^a{}_{[cb,d]} \tag{3.39}$$

$$R^a{}_{b[cd;e]} = -2\, R^a{}_{bf[d}\, \Gamma^f{}_{ec]} \ . \tag{3.40}$$

The geodesic deviation equation is modified by the torsion. Setting it up in the same way as before, for vectorfields k^a, p^a:

$$k^c\, k^b{}_{;c} = 0 \tag{3.41}$$

$$\mathcal{L}_{(k)}\, p^a = 0 \tag{3.42}$$

$$\begin{aligned} k^b\, k^c\, p^a{}_{;bc} &= R^a{}_{bcd}\, k^b\, k^c\, p^d + \Gamma^a{}_{[bc];d}\, l^b\, k^d\, p^c \\ &+ \Gamma^a{}_{[bc]}\, k^b\, k^d\, p^c{}_{;d} \end{aligned} \tag{3.43}$$

We can rewrite eq.(3.43) several ways, none of them particularly simple. On the basis of the geodesic deviation equation in general relativity it can be shown that $R^a{}_{bcd}$ is completely determinable by measurement of the relative acceleration of nearby test particles. With the added complication in its replacement, eq.(3.43), it may be difficult to separate the

curvature from the torsion on the basis of physical measurements.

Hermitian Metric

The hermitian formulation involves the assumption that the metric is hermitian symmetric:

$$g_{ab} = \bar{g}_{ba} \tag{3.44}$$

(the bar indicates complex conjugation). This means $g_{(ab)}$ is real, $g_{[ab]}$ imaginary. The easiest way to work with this is to simply take the theory's new constant to be imaginary, ie, ik, instead of real. This formulation gives the metric a symmetry again, and results in the unusual "regular" solution (eq.2.18) already discussed, but the complexification that necessarily results is difficult to interpret.

From field equation (2.2), relating g_{ab} and Γ^a_{bc}, for a diagonal component of the metric g_{aa} (no sum), which is real:

$$\begin{aligned} g_{aa,c} &= g_{ad}\,\Gamma^d_{ca} + g_{da}\,\Gamma^d_{ac} \\ &= g_{ad}\,\Gamma^d_{ca} + \bar{g}_{ad}\,\Gamma^d_{ac}\,. \end{aligned} \tag{3.45}$$

The two terms on the right hand side must be complex conjugates. This implies:

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{cb}\,, \tag{3.46}$$

the connection is hermitian in its lower indices. It is interesting to note that the other connection L^a_{bc} is not so nicely hermitian symmetric. Using the projective transformation (2.43), we find that $L^a_{[bc]}$ is pure imaginary, but that $L^a_{(bc)}$ generally has both real and imaginary parts.

Writing out the geodesic equation:

$$\begin{aligned} k^c\,k^a{}_{;c} &= k^c\,k^a{}_{,c} + \Gamma^a_{bc}\,k^b\,k^c \\ &= 0 \end{aligned} \tag{3.47}$$

we see that only $\Gamma^a_{(bc)}$ is involved, so this is a real differential equation. With real initial data it will generate real solutions (geodesics in

real space). The general parallel transport equation:

$$\begin{aligned} t^c k^a{}_{;c} &= t^c k^a{}_{,c} + \Gamma^a_{bc}\, k^b\, t^c \\ &= 0 \end{aligned} \tag{3.48}$$

involves the full Γ^a_{bc} and thus is a differential equation with complex coefficients, so for real initial data for k^a we must expect the solution to be complex, even if transport is along a real t^c.

Moffat has suggested(39) using complex conjugation in the inner product in this formulation because it is then preserved in parallel transport (under certain circumstances), eg:

$$g_{ab}\, k^a\, \bar{t}^b\ . \tag{3.49}$$

With this inner product the magnitude of a vector is always real, but the inner product of two vectors is complex, and ambiguous: the complex conjugate:

$$\overline{(g_{ab}\, k^a\, \bar{t}^b)} = g_{ab}\, t^a\, \bar{k}^b \tag{3.50}$$

is just as likely a choice (we will see that the other possibilities, $g_{ab}\, \bar{t}^a\, k^b$ and its conjugate, do not mesh with parallel propagation as nicely). We can generalize the concept: to define the trace of a tensor(59) $Q^{a\ldots i\ldots j\ldots}{}_{b\ldots}$ on some two upper indices i, j, (there is similar argument for two lower) we must resolve the tensor into vectors. There exist $A^a_{(n)},\ldots,I^i_{(n)},\ldots,J^j_{(n)},\ldots,B^{(n)}_b,\ldots$ such that:

$$Q^{a\ldots i\ldots j\ldots}{}_{b\ldots} = \sum_n A^a_{(n)}\ldots I^i_{(n)}\ldots J^j_{(n)}\ldots B^{(n)}_b\ldots\ . \tag{3.51}$$

Then (being careful not to use a misleading shorthand notation for the trace):

$$g_{ij} Q^{a\ldots i\ldots \bar{j}\ldots}{}_{b\ldots} = g_{ij} \sum_n A^a_{(n)}\ldots I^i_{(n)}\ldots \bar{J}^j_{(n)}\ldots B^{(n)}_b\ldots\ . \tag{3.52}$$

Let us parallel propagate k^a and t^a (not $\bar{t}^a$) along m^a:

$$m^c\, k^a{}_{;c} = 0 \tag{3.53}$$

$$m^c\, t^a{}_{;c} = 0 \tag{3.54}$$

$$m^c(g_{ab}\, k^a\, \bar{t}^b)_{;c} = m^c\, g_{ab;c}\, k^a\, \bar{t}^b + m^c\, g_{ab}\, k^a{}_{;c}\bar{t}^b$$

$$+ m^c\, g_{ab} k^a(\bar{t}^{\,b})_{;c} \tag{3.55}$$

$$= k^a(m^c\, \bar{t}^b\, g_{ab,c} - m^c\bar{t}^b\, g_{ad}\, \Gamma^d_{bc} - m^c\, \bar{t}^b\, g_{db}\, \Gamma^d_{ac}$$

$$+ m^c\, g_{ab}\, \bar{t}^b{}_{,c} + m^c\, g_{ab}\, \bar{t}^d\, \Gamma^b_{dc})$$

Taking the complex conjugate of (3.54):

$$\bar{m}^c\, \bar{t}^b{}_{,c} = -\, \bar{m}^c\, \bar{t}^d\, \Gamma^b_{cd}\ . \tag{3.56}$$

If m^c is real, then:

$$m^c(g_{ab}\, k^a\, \bar{t}^b)_{;c} = k^a(m^c\, \bar{t}^b\, g_{ab,c} - m^c\, \bar{t}^b\, g_{db}\, \Gamma^d_{ac}$$

$$- m^c\, g_{ab}\, \bar{t}^d\, \Gamma^b_{cd})$$

$$= k^a\, \bar{t}^b\, m^c(g_{ab,c} - g_{db}\, \Gamma^d_{ac} - g_{ad}\, \Gamma^d_{cb}\ ) \tag{3.57}$$

$$= 0$$

by virtue of the field equation (2.2). Thus the inner product is preserved in parallel transport along real vectors.

The major problem of the hermitian theory now confronts us. Just what does the increased information resulting from complexification represent? What is the significance of a complex valued inner product (complex valued angle?)? Does spacetime now have four complex dimensions (equivalent to eight real dimensions), so that complex vectors merely point in complex directions? If so, what is the physical significance of the increased dimensionality, only detectable when there is an electromagnetic field? Note that parallel transport does not preserve inner products in all directions: the metric is once again not compatible with the connection. Is it possible that spacetime still has just four (real) dimensions, and the complex part of a vector does not "point somewhere", but represents something else? (What?) Given a complex vector, how do we parallel transport along it, or find the geodesic

tangent to it? If we simply take its real part (itself a vector in this case), we find the length of the real part is not the same as that of the whole. A timelike complex vector can have null or spacelike real part. Thus the ~~preserved~~ inner product loses geometric meaning. Since we have no resolutions to these problems at the present time, we will return to working with a real metric.

Killing Vectorfields

When we dicussed Killing vectorfields in general relativity, we identified them as vectorfields (eg k^a) along which the Lie derivative of the metric vanishes (eq.(3.21)):

$$\mathcal{L}_{(k)}\, g_{(ab)} = 0 \ ,$$

$g_{(ab)}$ representing all the gravitational information. With an electromagnetic field present we would add:

$$\mathcal{L}_{(k)}\, F_{ab} = 0 \ . \tag{3.58}$$

In nonsymmetric field theories, with g_{ab} representing all the gravitational and electromagnetic information, and with the Lie dervative existing independent of metric and connection, the obvious choice is:

$$\mathcal{L}_{(k)}\, g_{ab} = 0 \ . \tag{3.59}$$

This can be written as (53):

$$\begin{aligned} 0 &= g_{ab;c}\, k^c + g_{ac}\, k^c{}_{;b} + g_{cb}\, k^c{}_{;a} \\ &= g_{ab,c}\, k^c + g_{ac}\, k^c{}_{,b} + g_{cb}\, k^c{}_{,a} \ . \end{aligned} \tag{3.60}$$

Using the field equation (2.2) to eliminate derivatives of g_{ab}, we can rewrite these in two alternate forms:

$$\begin{aligned} 0 &= g_{ac}\, k^c{}_{;b} + g_{cb}\, k^c{}_{;a} + 2g_{cb}\, \Gamma^c_{[ad]} k^d \\ &= g_{ac}\, k^c{}_{,b} + g_{cb}\, k^c{}_{,a} + g_{ac} \Gamma^c_{db}\, k^d + g_{cb} \Gamma^c_{ad}\, k^d \ . \end{aligned} \tag{3.61}$$

In analogy with general relativity consider the value of the inner product of k^a with t^a, the tangent to a geodesic, along the geodesic:

$$t^c(g_{ab}\, k^a\, t^b)_{;c} = g_{ab;c}\, k^a\, t^b\, t^c + g_{ab}\, k^a{}_{;c}\, t^b\, t^c$$
$$= g_{ab}\, t^b\, t^c\, k^a{}_{;c} \tag{3.62}$$

$$t^c(g_{ab}\, t^a\, k^b)_{;c} = g_{ab;c}\, t^a\, k^b\, t^c + g_{ab}\, t^a\, k^b{}_{;c}\, t^c \quad . \tag{3.63}$$

Neither choice of inner product is preserved nor does there appear to be any combination of them which is preserved. Defining the inner product with $g_{(ab)}$ does not improve the situation (unless we also use $\Gamma^a_{bc(bc)}$). Killing vector fields do not appear to generate constants of motion in any simple manner.

The Viability of the Theory

In this chapter we have seen a large number of problems presently associated with BMB theory, most of them under the topic of finding physical meaning for the geometric structure, that is, translating the mathematical theory into the physics we are really interested in (beyond the minimal physics. Since timelike, null and spacelike geodesics exist, we can identify paths of massive and massless freely-falling particles). Until most of these problems are resolved BMB theory will be at a disadvantage when compared to EMT because we do not know how the mathematics translates into physical prediction. We will not just abandon the theory, because there are some areas of study immediately obvious which may provide solutions to some of these problems. One of them is geometric optics: by investigating light and gravitational radiation propagation we should be able to see which connection really is used by null fields. This will be discussed in Chapter 5.

Chapter IV

Perturbation Expansions

Perturbations in EMT

Perturbation theory is a well known approximation method in physics, including general relativity. On some initial configuration (assumed known), we consider a change in the field quantities which is small enough to produce only small changes in all other relevant quantities. Then the field equations are expanded in the small amplitude of the change, λ, and each order of "smallness" is solved individually. With a linear theory we can formally write out a power series in λ and declare that there is some finite radius of convergence in λ. With a nonlinear theory (both EMT and BMB are nonlinear), convergence is harder to assess. Isaacson[60] indicates that possibly only the first (linear) correction is to be trusted.

We will write out expressions for EMT perturbations here (this is all fairly well known material) so that we can compare them with their BMB analogues. There are three cases we will be concerned with, in order of increasing complexity: empty Minkowski space background, gravitational but no electromagnetic field background, and general background fields.

Formally, we assume all relevant quantities can be expanded into (at least) a polynomial in λ with possible other terms in higher orders:

$$g_{ab} = \overset{o}{g}_{ab} + \lambda \overset{'}{g}_{ab} + \lambda^2 \overset{''}{g}_{ab} + \ldots \tag{4.1}$$

$$\Gamma^{a}_{\ bc} = \overset{o}{\Gamma}{}^{a}_{\ bc} + \lambda \overset{'}{\Gamma}{}^{a}_{\ bc} + \lambda^2 \overset{''}{\Gamma}{}^{a}_{\ bc} + \ldots \tag{4.2}$$

$$F_{ab} = \overset{o}{F}_{ab} + \lambda \overset{'}{F}_{ab} + \lambda^2 \overset{''}{F}_{ab} + \ldots \quad . \tag{4.3}$$

This implies, for example:

$$R_{ab} = \overset{o}{R}_{ab} + \lambda \overset{'}{R}_{ab} + \lambda^2 \overset{''}{R}_{ab} + \ldots \quad . \tag{4.4}$$

λ is an artificial, dimensionless "small parameter":

$$\lambda \ll 1 \tag{4.5}$$

which is in fact eventually ignored, but it is convenient when handling

several independent types of smallness (each can be labelled differently), as will be necessary later. We take all quantities in the equations other than λ as being of order unity. The order zero (in λ) quantities are assumed to be those of some exact solution in EMT (later, in BMB theory).

With empty Minkowski background (first order will be linearized theory):

$$\overset{o}{g}_{ab} = \eta_{ab} \tag{4.6}$$

$$\overset{o}{R}_{ab} = 0 \ . \tag{4.7}$$

Necessarily,

$$\overset{o}{F}_{ab} = 0 \tag{4.8}$$

and, for simplicity, we choose the obvious coordinate condition:

$$\overset{o}{\Gamma}{}^{a}{}_{bc} = 0 \ . \tag{4.9}$$

The general EMT field equations are:

$$R_{ab} = 8\pi T_{ab} \tag{4.10}$$

$$F_{[ab,c]} = 0 \tag{4.11}$$

$$F^{ab}{}_{;b} = 0 \tag{4.12}$$

$$T_{ab} = F_a{}^c F_{cb} + \tfrac{1}{4} g_{ab} F_{cd} F^{cd} \ . \tag{4.13}$$

Applying our expansions in this case, we find (after noting that the zeroth order is trivial) that T_{ab} is of order λ^2, being quadratic in F_{ab}, and that to first order covariant differentiation is coordinate differentiation. Then the first order gravitational and electromagnetic equations decouple:

$$\overset{'}{\Gamma}{}^{a}{}_{bc} = \tfrac{1}{2}\eta^{ad}(\overset{'}{g}_{bd,c} + \overset{'}{g}_{dc,b} - \overset{'}{g}_{bc,d}) \tag{4.14}$$

$$\overset{'}{R}_{ab} = \overset{'}{\Gamma}{}^{c}{}_{ab,c} - \overset{'}{\Gamma}{}^{c}{}_{ac,b} = 0 \tag{4.15}$$

$$\overset{'}{F}_{[ab,c]} = 0 \tag{4.16}$$

$$\overset{'}{F}{}^{ab}{}_{,b} = 0 \ , \tag{4.17}$$

and the first order gravitational and electromagnetic perturbations are independent of each other. If we go to second order (in spite of Issacson's warning) we see that $\overset{''}{R}_{ab}$ is modified by $\overset{'}{F}_{ab}$ through $\overset{''}{T}_{ab}$, and $\overset{''}{F}_{ab}$ is modified by $\overset{'}{g}_{ab}$ through $\overset{'}{\Gamma}{}^a_{bc}$ appearing in covariant differentiation.

In the case of a purely gravitational background we have nontrivial $\overset{o}{g}_{ab}$, $\overset{o}{\Gamma}{}^a_{bc}$, and $\overset{o}{R}_{ab}$ but retain zero $\overset{o}{F}_{ab}$. Now, order zero is not trivial but can be anticipated:

$$\overset{o}{R}_{ab} = 0 \ . \tag{4.18}$$

The first order equations (T_{ab} is again second order) are:

$$\begin{aligned}\overset{'}{\Gamma}{}^a_{bc} &= \tfrac{1}{2}\, \overset{'}{g}{}^{ad} \left(\overset{o}{g}_{bd,c} + \overset{o}{g}_{dc,b} - \overset{o}{g}_{bc,d}\right)\\ &+ \tfrac{1}{2}\, \overset{o}{g}{}^{ad} \left(\overset{'}{g}_{bd,c} + \overset{'}{g}_{dc,b} - \overset{'}{g}_{bc,d}\right)\end{aligned} \tag{4.19}$$

$$\begin{aligned}\overset{'}{R}_{ab} &= \overset{'}{\Gamma}{}^c_{ab,c} - \overset{'}{\Gamma}{}^c_{ac,b} - \overset{o}{\Gamma}{}^c_{ad}\, \overset{'}{\Gamma}{}^d_{cb} - \overset{o}{\Gamma}{}^d_{cb}\, \overset{'}{\Gamma}{}^c_{ad}\\ &+ \overset{o}{\Gamma}{}^c_{dc}\, \overset{'}{\Gamma}{}^d_{ab} + \overset{o}{\Gamma}{}^d_{ab}\, \overset{'}{\Gamma}{}^c_{dc}\\ &= 0\end{aligned} \tag{4.20}$$

$$\overset{'}{F}{}^{ab}{}_{;b} = 0 \tag{4.21}$$

$$\overset{'}{F}_{[ab,c]} = 0 \tag{4.22}$$

$\overset{o}{g}{}^{ab}$ is the inverse of $\overset{o}{g}_{ab}$, $\overset{'}{g}{}^{ab}$ is defined by the expression:

$$g^{ab} = \overset{o}{g}{}^{ab} + \lambda \overset{'}{g}{}^{ab} + \ldots \tag{4.23}$$

and can be written:

$$\overset{'}{g}{}^{ab} = - \overset{o}{g}{}^{ac}\, \overset{o}{g}{}^{bd}\, \overset{'}{g}_{cd} \ . \tag{4.24}$$

In eq. (4.21), the notation "$\underset{o}{;}$" means covariant derivative with respect to $\overset{o}{\Gamma}{}^a_{bc}$, as the divergence equation is only taken to first order (the indices have been raised with $\overset{o}{g}{}^{ab}$). The gravitational and electromagnetic disturbances here are again independent to first order, and in second order $\overset{''}{R}_{ab}$ is modified by $\overset{'}{F}_{ab}$ through $\overset{''}{T}_{ab}$.

Finally, in the case of a fully general background we have the standard EMT equations for order zero:

$$\overset{\circ}{R}_{ab} = 8\pi \overset{\circ}{T}_{ab} \tag{4.25}$$

$$\overset{\circ}{F}{}^{ab}{}_{\underset{\circ}{;}b} = 0 \tag{4.26}$$

$$\overset{\circ}{F}_{[ab,c]} = 0 \tag{4.27}$$

and T_{ab} has a first order part:

$$\overset{\prime}{T}_{ab} = \overset{\prime}{g}{}^{cd} \overset{\circ}{F}_{ac} \overset{\circ}{F}_{db} + \overset{\circ}{F}_{a}{}^{c} \overset{\prime}{F}_{cb} + \overset{\circ}{F}{}^{c}{}_{b} \overset{\prime}{F}_{ac} \tag{4.28}$$

$$+ \tfrac{1}{2} \overset{\circ}{g}_{ab} (\overset{\circ}{F}{}^{cd} \overset{\prime}{F}_{cd} + \overset{\prime}{g}{}^{cd} \overset{\circ}{F}{}^{e}{}_{c} \overset{\circ}{F}_{ed}) + \tfrac{1}{4} \overset{\prime}{g}_{ab} \overset{\circ}{F}{}^{cd} \overset{\circ}{F}_{cd} \quad ,$$

where indices of $\overset{\circ}{F}_{ab}$ have been raised with $\overset{\circ}{g}{}^{ab}$. Here $\overset{\prime}{g}{}^{ab}$ still has the form of eq. (4.24), and:

$$F^{ab} = g^{ac} g^{bd} F_{cd}$$

$$= \overset{\circ}{F}{}^{ab} + \lambda \overset{\prime}{F}{}^{ab} + \ldots \tag{4.29}$$

$$\overset{\prime}{F}{}^{ab} = \overset{\circ}{g}{}^{ac} \overset{\circ}{g}{}^{bd} \overset{\prime}{F}_{cd} + \overset{\circ}{g}{}^{ac} \overset{\prime}{g}{}^{bd} \overset{\circ}{F}_{cd} + \overset{\prime}{g}{}^{ac} \overset{\circ}{g}{}^{bd} \overset{\circ}{F}_{cd}. \tag{4.30}$$

$\overset{\prime}{R}_{ab}$ has the same form in Γ^{a}_{bc} as was seen in (4.20). The first order field equations are eq. (4.19) and:

$$\overset{\prime}{R}_{ab} = 8\pi \overset{\prime}{T}_{ab} \tag{4.31}$$

$$\overset{\prime}{F}{}^{ab}{}_{,b} + \overset{\prime}{F}{}^{ac} \overset{\circ}{\Gamma}{}^{b}_{cb} + \overset{\prime}{F}{}^{cb} \overset{\circ}{\Gamma}{}^{a}_{cb} + \overset{\circ}{F}{}^{ac} \overset{\prime}{\Gamma}{}^{b}_{cb} + \overset{\circ}{F}{}^{cb} \overset{\prime}{\Gamma}{}^{a}_{cb} = 0 \tag{4.32}$$

$$\overset{\prime}{F}_{[ab,c]} = 0 \quad . \tag{4.33}$$

The first three terms of eq(4.32) are just $\overset{\prime}{F}{}^{ab}{}_{\underset{\circ}{;}b}$. The detailed form of these equations is rather complicated and difficult to solve at present. Note, however, that a $\overset{\prime}{g}_{ab}$ usually generates an $\overset{\prime}{F}_{ab}$, and <u>vice versa</u>.

<u>Perturbations In BMB</u>

- Minkowski Background

For Minkowski background in BMB we take, in analogy with EMT:

$$g_{ab} = \eta_{ab} + \lambda \overset{\prime}{g}_{ab} + \lambda^2 \overset{\prime\prime}{g}_{ab} + \ldots \tag{4.34}$$

where η_{ab} is the flat (symmetric) Minkowski metric. Also:

$$g^{ab} = \eta^{ab} + \lambda \overset{\prime}{g}{}^{ab} + \lambda^2 \overset{\prime\prime}{g}{}^{ab} + \dots \,, \tag{4.35}$$

and by solving:

$$g^{ab} g_{cb} = \delta^a_c \tag{4.36}$$

we find:

$$\overset{\prime}{g}{}^{ab} = - \eta^{ac} \eta^{db} \overset{\prime}{g}_{dc} \tag{4.37}$$

$$\overset{\prime\prime}{g}{}^{ab} = - \eta^{ac} \eta^{db} \overset{\prime\prime}{g}_{dc} + \eta^{ac} \eta^{db} \eta^{ef} \overset{\prime}{g}_{ec} \overset{\prime}{g}_{df} \,. \tag{4.38}$$

Now:

$$\Gamma^a_{bc} = \lambda \overset{\prime}{\Gamma}{}^a_{bc} + \lambda^2 \overset{\prime\prime}{\Gamma}{}^a_{bc} + \dots \tag{4.39}$$

and (4.34) in field eq. (2.2) relating g_{ab} and Γ^a_{bc} imply:

$$\overset{\prime}{\Gamma}{}^a_{(bc)} = \tfrac{1}{2} \eta^{ad} (\overset{\prime}{g}_{(dc),b} + \overset{\prime}{g}_{(bd),c} - \overset{\prime}{g}_{(bc),d}) \tag{4.40}$$

$$\overset{\prime\prime}{\Gamma}{}^a_{(bc)} = \tfrac{1}{2} \eta^{ad} (\overset{\prime\prime}{g}_{(dc),b} + \overset{\prime\prime}{g}_{(bd),c} - \overset{\prime\prime}{g}_{(bc),d} - 2 \overset{\prime}{g}_{[ec]} \overset{\prime}{\Gamma}{}^e_{[db]} - 2 \overset{\prime}{g}_{[be]} \overset{\prime}{\Gamma}{}^e_{[cd]} - 2 \overset{\prime}{g}_{(de)} \overset{\prime}{\Gamma}{}^e_{(bc)}) \tag{4.41}$$

$$\overset{\prime}{\Gamma}{}^a_{[bc]} = \tfrac{1}{2} \eta^{ad} (\overset{\prime}{g}_{[dc],b} + \overset{\prime}{g}_{[bd],c} + \overset{\prime}{g}_{[bc],d}) \tag{4.42}$$

$$\overset{\prime\prime}{\Gamma}{}^a_{[bc]} = \tfrac{1}{2} \eta^{ad} (\overset{\prime\prime}{g}_{[dc],b} + \overset{\prime\prime}{g}_{[bd],c} + \overset{\prime\prime}{g}_{[bc],d} - 2 \overset{\prime}{g}_{[ec]} \overset{\prime}{\Gamma}{}^e_{(db)} - 2 \overset{\prime}{g}_{[be]} \overset{\prime}{\Gamma}{}^e_{(cd)} - 2 \overset{\prime}{g}_{(de)} \overset{\prime}{\Gamma}{}^e_{[bc]}) \,. \tag{4.43}$$

R_{ab} in terms of Γ^a_{bc} is:

$$\overset{\prime}{R}_{(ab)} = \overset{\prime}{\Gamma}{}^c_{(ab),c} - \tfrac{1}{2} (\overset{\prime}{\Gamma}{}^c_{(ac),b} + \overset{\prime}{\Gamma}{}^c_{(bc),a}) \tag{4.44}$$

$$\overset{\prime\prime}{R}_{(ab)} = \overset{\prime\prime}{\Gamma}{}^c_{(ab),c} - \tfrac{1}{2} (\overset{\prime\prime}{\Gamma}{}^c_{(ac),b} + \overset{\prime\prime}{\Gamma}{}^c_{(bc),a}) - \overset{\prime}{\Gamma}{}^c_{(ad)} \overset{\prime}{\Gamma}{}^d_{(cb)} - \overset{\prime}{\Gamma}{}^c_{[ad]} \overset{\prime}{\Gamma}{}^d_{[cb]} + \overset{\prime}{\Gamma}{}^d_{(cd)} \overset{\prime}{\Gamma}{}^c_{(ab)} \tag{4.45}$$

$$\overset{\prime}{R}_{[ab]} = \overset{\prime}{\Gamma}{}^c_{[ab],c} \tag{4.46}$$

$$\overset{\prime\prime}{R}_{[ab]} = \overset{\prime\prime}{\Gamma}{}^c_{[ab],c} - \overset{\prime}{\Gamma}{}^c_{(ad)} \overset{\prime}{\Gamma}{}^d_{[cb]} - \overset{\prime}{\Gamma}{}^c_{[ad]} \overset{\prime}{\Gamma}{}^d_{(cb)} + \overset{\prime}{\Gamma}{}^d_{(cd)} \overset{\prime}{\Gamma}{}^c_{[ab]} \,. \tag{4.47}$$

These last expressions are written using the tracelessness of the torsion

(which here becomes tracelessness in each order). I_{ab} in terms of g_{ab} is:

$$\overset{'}{I}_{(ab)} = 0 \tag{4.48}$$

$$\overset{''}{I}_{(ab)} = \frac{-8\pi}{k^2}\left(\eta^{cd}\,\overset{'}{g}_{[ad]}\overset{'}{g}_{[cb]} + \tfrac{1}{4}\eta_{ab}\,\overset{'}{g}_{[cd]}\overset{'}{g}^{[cd]}\right) \tag{4.49}$$

$$\overset{'}{I}_{[ab]} = \frac{-8\pi}{k^2}\overset{'}{g}_{[ab]} \tag{4.50}$$

$$\overset{''}{I}_{[ab]} = \frac{-8\pi}{k^2}\overset{''}{g}_{[ab]} \; . \tag{4.51}$$

From these the field equations in first order (after solving for $\overset{'}{\Gamma}{}^{a}_{bc}$) are:

$$\eta^{bc}\,\overset{'}{g}_{[ba],c} = 0 \tag{4.52}$$

$$\tfrac{1}{2}\Box^2\,\overset{'}{g}_{[ab]} - \frac{8\pi}{k^2}\overset{'}{g}_{[ab]} = \overset{'}{V}_{a,b} - \overset{'}{V}_{b,a} \tag{4.53}$$

$$-\Box^2\overset{'}{g}_{(ab)} + \eta^{cd}(-\overset{'}{g}_{(cd),ab} + \overset{'}{g}_{(db),ac} + \overset{'}{g}_{(ca),db}) = 0 \; . \tag{4.54}$$

Note that in first order, the gravitational and electromagnetic perturbations ($\overset{'}{g}_{(ab)}$ and $\overset{'}{g}_{[ab]}$) are independent. Eq.(4.54) is one that can be found in EMT, meaning first order gravitational perturbations are the same as EMT's. The second order general case equations are complicated but we can look at special cases(see Appendix). If we set $\overset{'}{g}_{[ab]}$ to zero, the gravitational and electromagnetic equations become independent down to second order, and the second order gravitational equations are equivalent to the analogous equations in EMT. The gravitational field does not generate an electromagnetic field unless there is an electromagnetic background to perturb (the same is true in EMT). If we set $\overset{'}{g}_{(ab)}$ to zero, then the second order symmetric field is governed by the equations of EMT plus terms of order k^2:

$$\overset{''}{R}_{ab(emt)} = 8\pi\,\overset{''}{T}_{ab(emt)} + O(k^2) \tag{4.55}$$

The most interesting solutions in this approximation are plane waves, representing radiation. They will be discussed in chapter 5.

- Vacuum Gravitational Background

For perturbations on a purely gravitational background:

$$g_{ab} = \overset{o}{g}_{(ab)} + \lambda \overset{'}{g}_{ab} + \ldots \tag{4.56}$$

$$\Gamma^a_{bc} = \overset{o}{\Gamma}{}^a_{(bc)} + \lambda \overset{'}{\Gamma}{}^a_{bc} + \ldots \tag{4.57}$$

$$R_{ab} = \overset{o}{R}_{(ab)} + \lambda \overset{'}{R}_{ab} + \ldots \quad . \tag{4.58}$$

Because of increasing complication, we will not look at the second order in anything but the Minkowski background case. BMB reduces to EMT when there is no $g_{[ab]}$, so we can correctly predict that the zeroth order solution is a vacuum general relativity geometry. Inspection of the first order equations shows that, in analogy with EMT, the gravitational ($\overset{'}{g}_{(ab)}$) and electromagnetic ($\overset{'}{g}_{[ab]}$) fields are completely independent in first order, so we can consider them separately. The equations in $\overset{'}{g}_{(ab)}$ are equivalent to those found in EMT (in fact general relativity without F_{ab}), and we need not write them out again. In $\overset{''}{g}_{[ab]}$ we find:

$$\overset{'}{\Gamma}{}^a_{[bc]} = \tfrac{1}{2} \overset{o}{g}{}^{(ad)} \left(\overset{'}{g}_{[bd],c} + \overset{'}{g}_{[dc],b} + \overset{'}{g}_{[bc],d} - 2\overset{'}{g}_{[be]} \overset{o}{\Gamma}{}^e_{(cd)} - 2\overset{'}{g}_{[ec]} \overset{o}{\Gamma}{}^e_{(db)}\right) . \tag{4.59}$$

The equation demanding zero trace:

$$\overset{'}{\Gamma}{}^c_{[ac]} = 0 \tag{4.60}$$

is then:

$$\overset{o}{g}{}^{(cd)} \left(\overset{'}{g}_{[ad],c} - \overset{'}{g}_{[ae]} \overset{o}{\Gamma}{}^e_{(dc)} - \overset{'}{g}_{[ed]} \overset{o}{\Gamma}{}^e_{(ac)}\right) = 0 \,. \tag{4.61}$$

The part in brackets is just $\overset{'}{g}_{[ad]\overset{;}{o}c}$, so this is a special statement of the divergence equation on $g_{[ab]}$ mentioned before, this time raising an index instead of using $g^{[ab]}$. For I_{ab} we find simply:

$$\overset{'}{I}_{(ab)} = 0 \tag{4.62}$$

$$\overset{'}{I}_{[ab]} = -\frac{8\pi}{k^2} \overset{'}{g}_{[ab]} \, . \tag{4.63}$$

The equation in $\overset{'}{R}_{[ab]}$ can be written:

$$
{}_{[}\Big(\overset{\prime}{\Gamma}{}^{c}_{[ab],c} - \overset{\circ}{\Gamma}{}^{c}_{(ad)}\overset{\prime}{\Gamma}{}^{d}_{[cb]} + \overset{\circ}{\Gamma}{}^{c}_{(bd)}\overset{\prime}{\Gamma}{}^{d}_{[ca]}
+ \overset{\circ}{\Gamma}{}^{d}_{(cd)}\overset{\prime}{\Gamma}{}^{c}_{[ab]} - \frac{8\pi}{k^2}\overset{\prime}{g}_{[ab]}\Big)_{,e]} = 0 \,, \tag{4.64}
$$

but in terms of g_{ab} alone it is complicated. These equations are used later in this chapter in finding BMB perturbations of the Schwarzschild geometry, and in chapter 5.

- General Background

In the general case the background $\overset{\circ}{g}_{ab}$ and $\overset{\circ}{\Gamma}{}^{a}_{bc}$ have symmetric and antisymmetric parts. It is no longer easy to solve for $\overset{\prime}{\Gamma}{}^{a}_{bc}$ in the straightforward manner used earlier, as both $\overset{\prime}{\Gamma}{}^{a}_{(bc)}$ and $\overset{\prime}{\Gamma}{}^{a}_{[bc]}$ appear in each of the equations on $\overset{\prime}{g}_{(ab)}$ and $\overset{\prime}{g}_{[ab]}$ (from eq.(2.2)):

$$
\begin{aligned}
0 = {} & \overset{\prime}{g}_{(ab),c} - \overset{\prime}{g}_{(ad)}\overset{\circ}{\Gamma}{}^{d}_{(bc)} - \overset{\circ}{g}_{(ad)}\overset{\prime}{\Gamma}{}^{d}_{(bc)} - \overset{\prime}{g}_{(db)}\overset{\circ}{\Gamma}{}^{d}_{(ac)} \\
& - \overset{\circ}{g}_{(db)}\overset{\prime}{\Gamma}{}^{d}_{(ac)} - \overset{\prime}{g}_{[ad]}\overset{\circ}{\Gamma}{}^{d}_{[cb]} - \overset{\circ}{g}_{[ad]}\overset{\prime}{\Gamma}{}^{d}_{[cb]} - \overset{\prime}{g}_{[db]}\overset{\circ}{\Gamma}{}^{d}_{[ac]} \\
& - \overset{\circ}{g}_{[db]}\overset{\prime}{\Gamma}{}^{d}_{[ac]}
\end{aligned} \tag{4.65}
$$

$$
\begin{aligned}
0 = {} & \overset{\prime}{g}_{[ab],c} - \overset{\prime}{g}_{[ad]}\overset{\circ}{\Gamma}{}^{d}_{(bc)} - \overset{\circ}{g}_{[ad]}\overset{\prime}{\Gamma}{}^{d}_{(bc)} - \overset{\prime}{g}_{[db]}\overset{\circ}{\Gamma}{}^{d}_{(ac)} \\
& - \overset{\circ}{g}_{[db]}\overset{\prime}{\Gamma}{}^{d}_{(ac)} - \overset{\prime}{g}_{(ad)}\overset{\circ}{\Gamma}{}^{d}_{[cb]} - \overset{\circ}{g}_{(ad)}\overset{\prime}{\Gamma}{}^{d}_{[cb]} - \overset{\prime}{g}_{(db)}\overset{\circ}{\Gamma}{}^{d}_{[ac]} \\
& - \overset{\circ}{g}_{(db)}\overset{\prime}{\Gamma}{}^{d}_{[ac]} \,.
\end{aligned} \tag{4.66}
$$

Of course we have:

$$
\overset{\prime}{\Gamma}{}^{c}_{[ac]} = 0. \tag{4.67}
$$

R_{ab} can be written:

$$
\begin{aligned}
\overset{\prime}{R}_{(ab)} = {} & \overset{\prime}{\Gamma}{}^{c}_{(ab),c} - \tfrac{1}{2}\big(\overset{\prime}{\Gamma}{}^{c}_{(ac),b} + \overset{\prime}{\Gamma}{}^{c}_{(bc),a}\big) - \overset{\circ}{\Gamma}{}^{c}_{(ad)}\overset{\prime}{\Gamma}{}^{d}_{(cb)} \\
& - \overset{\prime}{\Gamma}{}^{c}_{(ad)}\overset{\circ}{\Gamma}{}^{d}_{(cb)} + \overset{\circ}{\Gamma}{}^{c}_{[ad]}\overset{\prime}{\Gamma}{}^{d}_{[cb]} + \overset{\prime}{\Gamma}{}^{c}_{[ad]}\overset{\circ}{\Gamma}{}^{d}_{[cb]} \\
& + \overset{\circ}{\Gamma}{}^{c}_{(ab)}\overset{\prime}{\Gamma}{}^{d}_{(cd)} + \overset{\prime}{\Gamma}{}^{c}_{(ab)}\overset{\circ}{\Gamma}{}^{d}_{(cd)}
\end{aligned} \tag{4.68}
$$

$$
\begin{aligned}
\overset{\prime}{R}_{[ab]} = {} & \overset{\prime}{\Gamma}{}^{c}_{[ab],c} - \overset{\circ}{\Gamma}{}^{c}_{(ad)}\overset{\prime}{\Gamma}{}^{d}_{[cb]} - \overset{\prime}{\Gamma}{}^{c}_{(ad)}\overset{\circ}{\Gamma}{}^{d}_{[cb]} - \overset{\circ}{\Gamma}{}^{c}_{[ad]}\overset{\prime}{\Gamma}{}^{d}_{(cb)} \\
& - \overset{\prime}{\Gamma}{}^{c}_{[ad]}\overset{\circ}{\Gamma}{}^{d}_{(cb)} + \overset{\circ}{\Gamma}{}^{c}_{[ab]}\overset{\prime}{\Gamma}{}^{d}_{(cd)} + \overset{\prime}{\Gamma}{}^{c}_{[ab]}\overset{\circ}{\Gamma}{}^{d}_{(cd)} \,.
\end{aligned} \tag{4.69}
$$

$I_{(ab)}$ and $I_{[ab]}$, when written out in terms of our expansion, are complicated expressions, because most of I_{ab} is cubic in g_{ab}, and includes the upper indexed metric:

$$\overset{'}{g}{}^{ab} = - \overset{o}{g}{}^{ac} \overset{o}{g}{}^{db} \overset{'}{g}_{dc} \tag{4.70}$$

(the symmetric and skew parts of this both have several terms). The field equations in contracted curvature are then very complicated. We will not actually deal with them until chapter 5, when the geometric optics approximation allows further simplification.

BMB Perturbation of the Schwarzschild Geometry.

In EMT, consider perturbing the Schwarzschild geometry with an electromagnetic field (to first order, any gravitational perturbation is independent, so we can ignore it. We are particularly interested in a possible BMB magnetic monopole in this section). We need only solve the electromagnetic equations (4.21), (4.22), and it is easy to show that they are satisfied by either a radial electric field:

$$E_r = \frac{Q}{r^2} \tag{4.71}$$

and/or a radial magnetic field:

$$B_r = \frac{L}{r^2} \tag{4.72}$$

(we are not ruling out EMT magnetic charge a priori here). Thus, the electric and magnetic Reissner-Nordstrom solution (eq. (2.62), (2.63)) is found in approximation. By doing analogous work in BMB theory we may find the known electic monopole, and some clue to a magnetic monopole, in approximation. Recall that in chapter 2, we said that no magnetic monopole solution has been found in the theory. If it has no such monopoles, then it would have an advantage over EMT, since EMT does not really preclude monopoles yet none have been found experimentally. The methods here constitute an independent test for monopoles. Any

approximate forms found will give indication of forms to look for in further exact calculations.

In BMB let us begin with a strictly radial electric field perturbation:

$$\overset{\prime}{g}_{[10]} = k\,E(r) \qquad (4.73)$$

in the usual (exterior) spherical coordinates. The background, Schwarzschild's solution, is well known:

$$ds^2 = (1 - \frac{2m}{r})\,dt^2 - \frac{dr^2}{(1 - \frac{2m}{r})} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (4.74)$$

and we will not write out the expressions for its geometric quantities. In straightforward solution, eq. (4.59) implies:

$$\overset{\prime}{\Gamma}{}^{1}_{[01]} = (1 - \frac{2m}{r})\,\frac{dE}{dr} \qquad (4.75)$$

$$\overset{\prime}{\Gamma}{}^{2}_{[02]} = (\frac{1}{r} - \frac{2m}{r^2})E \qquad (4.76)$$

$$= \overset{\prime}{\Gamma}{}^{3}_{[03]} \quad ,$$

Then eq. (4.60) implies:

$$\frac{dE}{dr} + \frac{2E}{r} = 0, \qquad (4.77)$$

the solution of which is:

$$E = \frac{Q}{r^2} \qquad (4.78)$$

(Q a constant of integration). Eq. (4.64) is then satisfied identically.

For the case of a radial magnetic field:

$$\overset{\prime}{g}_{[23]} = B(r)\,r^2 \sin\theta \quad . \qquad (4.79)$$

Solving eq. (4.59) we find:

$$\overset{\prime}{\Gamma}{}^{1}_{[23]} = -\tfrac{1}{2}\,(1 - \frac{2m}{r})\left[\frac{dB}{dr}\;r^2\sin\theta \;-\; 2Br\sin\theta\right] . \qquad (4.80)$$

$$\overset{\prime}{\Gamma}{}^{2}_{[13]} = -\tfrac{1}{2}\,\frac{dB}{dr}\;\sin\theta \qquad (4.81)$$

$$\overset{\prime}{\Gamma}{}^{3}_{[12]} = \frac{1}{2\sin\theta}\;\frac{dB}{dr} \quad . \qquad (4.82)$$

Eq. (4.60) is trivial, as the torsion's trace is identically zero. In eq. (4.64), let us condider the various combinations of the three free indices in:

$$(\dot{R}_{[ab]} + \dot{I}_{[ab]})_{,c} \tag{4.83}$$

before antisymmetrizing. It is easy to verify that if any index is zero, the expression is identically zero. Then if the last index is three, the expression is zero, as the term to be differentiated is independent of ϕ. This leaves two independent cases, (a,b,c) equal (3,1,2) or (2,3,1). In the first case:

$$\begin{aligned}(\dot{R}_{[31]} + \dot{I}_{[31]})_{,2} &= (\dot{\Gamma}^{2}_{[31],2} - \overset{o}{\Gamma}{}^{2}_{(33)} \dot{\Gamma}^{3}_{[21]} \\ &\quad - \overset{o}{\Gamma}{}^{3}_{(32)} \dot{\Gamma}^{2}_{[31]} + \overset{o}{\Gamma}{}^{3}_{(23)} \dot{\Gamma}^{2}_{[31]})_{,2} \\ &= \left(\left(\tfrac{1}{2} \frac{dB}{dr} \sin\theta\right)_{,2} - (-\sin\theta\cos\theta)\left(\frac{-1}{2\sin\theta}\frac{dB}{dr}\right) \right)_{,2} \\ &= 0 ,\end{aligned} \tag{4.84}$$

and in the second:

$$\begin{aligned}(\dot{R}_{[23]} + \dot{I}_{[23]})_{,1} &= (\dot{\Gamma}^{1}_{[23],1} - \overset{o}{\Gamma}{}^{2}_{(21)} \dot{\Gamma}^{1}_{[23]} - \overset{o}{\Gamma}{}^{1}_{(22)} \dot{\Gamma}^{2}_{[13]} \\ &\quad + \overset{o}{\Gamma}{}^{3}_{(31)} \dot{\Gamma}^{1}_{[32]} + \overset{o}{\Gamma}{}^{1}_{(33)} \dot{\Gamma}^{3}_{[12]} + \overset{o}{\Gamma}{}^{d}_{(1d)} \dot{\Gamma}^{1}_{[23]} - \frac{8\pi}{k^2} \dot{g}_{[23]})_{,1} \\ &= \Big[-\frac{m}{r}\left(\frac{dB}{dr} r^2 \sin\theta - 2\,Br\sin\theta\right) \\ &\quad - \tfrac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{d^2B}{dr^2} r^2 \sin\theta - 2\,B\sin\theta\right) \\ &\quad - \left(1-\frac{2m}{r}\right)\frac{dB}{dr} r\sin\theta - \frac{8\pi}{k^2} Br^2 \sin\theta \Big]_{,1} \\ &= \left[\frac{d^2B}{dr^2}\left(-\frac{r^2}{2}\right)\left(1-\frac{2m}{r}\right) + \frac{dB}{dr}(m-r) + B\left(1-\frac{8\pi r^2}{k^2}\right) \right]_{,1} \sin\theta .\end{aligned} \tag{4.85}$$

Since there is only one nonzero term in eq.(4.83), antisymmetrization over the three indices has no effect, and eq.(4.64) simply means eq.(4.85) is zero:

$$\frac{d^2B}{dr^2}\left(-\frac{r^2}{2}\right)\left(1-\frac{2m}{r}\right) + \frac{dB}{dr}(m-r) + B\left(1-\frac{8\pi r^2}{k^2}\right) = K . \tag{4.86}$$

K is an integration constant; to interpret it we take the small k limit of eq.(4.86):

$$-\frac{8\pi r^2}{k^2} B = K \tag{4.87}$$

$$B = -\frac{K\,k^2}{8\pi r^2}$$

$$= \frac{L}{r^2} \qquad (4.88)$$

$$K = -\frac{8\pi L}{k^2}\ . \qquad (4.89)$$

Assuming a power series in r that satisfies the boundary condition:

$$\lim_{r\to\infty} B = 0\ , \qquad (4.90)$$

that is:

$$B(r) = \sum_{n=1}^{\infty} \frac{a_n}{r^n}\ , \qquad (4.91)$$

we find:

$$B(r) = \frac{L}{r^2} + \frac{k^2 mL}{2\pi r^5} + \ldots\ , \qquad (4.92)$$

where for the higher order terms ($n \geq 5$):

$$a_n = \frac{k^2}{8\pi}\left[\left(-\frac{n^2}{2} + \frac{5n}{2} - 2\right) a_{n-2} + (n-3)^2\, m\, a_{n-3}\right]. \qquad (4.93)$$

The series does not terminate to form a polynomial.

Several comments can be made about the series solution. Since it is an infinite series, it is not among the family of polynomial solutions tested for in exact calculation. The form of the recursion relation is complicated, so that the series is not recognizable as any familiar function, and we must worry about convergence. The fact that there are three parameters (k^2 , m,r) whose relative sizes are important and the complicated form of eq.(4.93) combine to make convergence very difficult to assess. We can note two limits. When k is zero, we get EMT, and just the expected first term. Also, all but the first term include a positive power of m, so that when m is zero we also find simply the first term. Only in these two limits do the electric and magnetic monopole have the same form(in this approximation). The m = 0 limit cannot be expected to be a physically interesting case. Experimentally,

electric charge is always accompanied by mass, yet in EMT's Reissner-Nordstrom solution, m, Q and L are completely independent. The massless charged Reissner-Nordstrom geometry is a naked singularity which is repulsive for neutral test masses. Only in the hermitian formulation of BMB might this case be physically tolerable. An infinite series magnetic field is difficult to work with in exact calculation for this case, so the general form found here may be difficult to actually verify or discredit. On the other hand, the simple form of the massless charge result suggests that one can test for it in exact calculation.

Chapter V

Radiation In The Geometric

Optics Approximation

Geometric Optics in EMT

- Introduction : Simple Cases

Elementary applications of geometric optics in general relativity are well known, and the more exotic cases we will eventually come to have been discussed in the literature[61-63]. By "geometric optics approximation" we mean the assumption of a periodic disturbance of high frequency, and expansion of the field equations into power series in the wavelength. In complicated cases, this is usually coupled with a weak field assumption to obtain equations simple enough to solve easily.

It is not always necessary to assume high frequency to find a reasonably physical radiative field[64]. The linearized EMT equations discussed in chapter 4 (eq.(4.14)-(4.16)) are well known to have plane wave solutions, gravitational and/or electromagnetic, which travel along null straight lines in Minkowski space.

We can, on the other hand, use high frequency without small amplitude as follows[65]. We assume there exists a geometry which, in some region, is a solution of the vacuum in EMT equations, eq.(4.10)-(4.13), where F_{ab} has the form:

$$F_{ab} = f_{ab} \exp\left(\frac{is}{\varepsilon}\right) + cc \tag{5.1}$$

(where "cc" indicates the complex conjugate of the preceding expression), that is, the electromagnetic field is strictly high frequency in the region. We assume that the Riemann tensor is bounded in the region. f_{ab} itself is slowly varying, but the exponential is of high frequency. ε is the small parameter (proportional to the reciprocal of frequency):

$$\varepsilon \ll 1 \tag{5.2}$$

in which expansions are made, and s is the "phase factor" whose derivative:

$$k_a = s_{;a} \tag{5.3}$$

$$= s_{,a}$$

is the propagation vector. Note that, as generated in eq.(5.3), the propagation vector is lower indexed, and is technically a "one-form" rather than a vector (upper indexed) pointing in four space[66]. The index must be raised with g^{ab} to make it the propagation vector(this distinction becomes significant in nonsymmetric theories).

The rapidly varying field here is in the standard form we will use throughout our study of geometric optics. It may be thought of in a Fourier analysis sense as a "bump" in the amplitude versus frequency function with central frequency $\frac{1}{\varepsilon}$ (but rigorous Fourier analysis is difficult to do in curved space). With nonlinear equations there tends to be interaction between different frequencies, and this is likely to be a significant interaction when there are both gravitational and electromagnetic waves in a region (if they are generated independently they are unlikely to have identical frequency), yet we will consistently use notation that considers only a single high frequency $(\frac{1}{\varepsilon})$ band. This is because we are using the geometric optics approximation to produce cases where the field equations are (approximately) linear, so that the interaction between different frequencies should not then be significant. In the cases in which we do find easy to solve equations, the results should be valid. In cases in which the approximate equations are complicated, we can conclude nothing beyond the fact that this approximation failed to linearize the situation.

Now, let us consider what form f_{ab} must have. The "curl" equation, eq.(4.11), implies the existence of a vector potential, A_b, which is high frequency:

$$\begin{aligned} A_b &= a_b \exp\left(\frac{is}{\varepsilon}\right) + cc \\ &= \tfrac{1}{2}\,\varepsilon\,\omega_b \exp\left(\frac{is}{\varepsilon}\right) + cc, \end{aligned} \tag{5.4}$$

(ω_b is the polarization vector) so that:

$$F_{ab} = i\omega_{[a}\, k_{b]} + \varepsilon\omega_{[a,b]} \; . \qquad (5.5)$$

Note that a_b is first order in ε(smallness), but it generates a unity order term in f_{ab} . Differentiation of a high frequency term of the nth order m times generates terms as large as ε^{n-m}, in general. Now, raising indices with g^{ab}, we find that the divergence equation, eq.(4.12), has a term of order $\frac{1}{\varepsilon}$:

$$F^{ab}{}_{;b} = (\exp(\tfrac{is}{\varepsilon})\; (\tfrac{-1}{\varepsilon})\, \omega^{[a}k^{b]}k_b + cc) + \; O(1) \qquad (5.6)$$

which (if nonzero) will be the dominant term in the high frequency limit (ie, as ε goes to zero), and which therefore must be zero. This implies:

$$k^b\, k_b = 0 \qquad (5.7)$$

$$\omega^a\, k_b = 0. \qquad (5.8)$$

The propagation vector is null: the high frequency disturbance propagates along null paths. Also, the polarization vector is orthogonal to the propagation vector.

Throughout this chapter we will be primarily concerned only with demonstrating the first property expected of "massless" radiation fields, that is, propagation along null geodesics. In general relativity we can use a small (but important for the geometric optics approximation) theorem: if the gradient of a scalar (eq.(5.3)) is null, then it is geodesic:

$$k^b\, k_{a;b} = 0 \quad . \qquad (5.9)$$

The proof is easy, but it relies on the covariant derivative of g^{ab} being zero and on the first two covariant derivatives of a scalar commuting (which is true with a symmetric connection. Neither of these properties is found in nonsymmetric theories). Thus we see that in an exact vacuum solution (region) with high frequency electromagnetic field, this radiative

field travels on null geodesics in the high frequency limit, and we are justified in identifying null cones with the light cones, in this type of situation.

-Gravitational Background

There is a slightly different problem than the one just discussed which is important in its own right. When we investigate the properties of some exact EMT (or BMB) solution, we look at paths of test particles and light, saying that we expect neutral test particles to follow timelike geodesics, and light to follow null geodesics, of the given solution. These expectations can only be trusted when supported by some calculations of how small, but not quite trivial, masses and small electromagnetic disturbances really propagate in such circumstances. We will not discuss the massive case here, since it is part of the theory of equations of motion, which we have not detailed[55], but we are in a position to discuss light propagation (In fact the framework here puts electromagnetic and gravitational disturbances on an equal footing, so we can discuss gravitational radiation, too).

Consider a physical light signal moving through some background, as a perturbation. If its amplitude is of order unity (as defined by the background), then its energy density will rival that of the background, causing significant changes in the gravitational geometry. Since we want small changes in the background, we must ask for small amplitude. Further, if its frequency is low, its wavelength is large, and in the extent of one wavelength, spacetime will be noticably curved. We can hardly expect the signal to travel geodesically if it does not see spacetime as locally flat, so we probably should consider only high frequencies[67]. These are physical arguments designed to explain the

fact that the field equations in this type of situation only become simple enough to solve easily after the application of both the weak field and geometric optics approximation.

Combining weak fields and geometric optics unfortunately appears to reduce mathematical rigour. We will expand all quantities into double series in independent parameters λ and ε. Much less is known about the convergence of such double series than about single parameter series. Furthermore, the order of taking the zero limits of the two parameters can be nontrivial. Classically, the energy density of an electromagnetic wave depends on the amplitude only, so to drive energy density to zero we must take λ to zero, regardless of ε. Further, the expressions found here tend to have unexpected properties when we take ε to zero while λ is finite, so we will assume that λ is not much greater than ε[68], symbolically:

$$\frac{\lambda}{\varepsilon} \lesssim 1 \quad . \tag{5.10}$$

Consider now small, high frequency electromagnetic and gravitational perturbations of a purely gravitational geometry. We already have applied the weak field approximation and found eq.(4.19)-(4.22). For high frequency within this we take:

$$g_{ab} = \overset{o}{g}_{ab} + \lambda g'_{ab} + O(\lambda^2) \tag{5.11}$$

$$g'_{ab} = (h_{ab}\exp(\frac{is}{\varepsilon}) + cc) + O(\varepsilon). \tag{5.12}$$

h_{ab} is slowly varying, of order unity, and we have not written out possible terms in g'_{ab} which are higher order in λ. Similarly:

$$F'_{ab} = f_{ab}\exp(\frac{is}{\varepsilon}) + cc + \ldots \quad . \tag{5.13}$$

Recall that the two types of perturbation are independent here: we can consider them separately, and even go back and use different s and ε for each.

The first order term in the perturbed connection $\lambda \overset{'}{\Gamma}{}^{a}_{bc}$ is now:

$$\lambda \overset{'}{\Gamma}{}^{a}_{bc} = \frac{\lambda}{\varepsilon}\Big[i \exp\left(\frac{is}{\varepsilon}\right) \overset{o}{g}{}^{ad}(h_{bd}\, k_c + h_{dc}\, k_b$$

$$- h_{bc}\, k_d) + cc\Big] + \ldots \tag{5.14}$$

$$= \frac{\lambda}{\varepsilon}\Big[\gamma^{a}_{bc} \exp\left(\frac{is}{\varepsilon}\right) + cc\Big] + \ldots \quad .$$

The leading term is of order $\frac{\lambda}{\varepsilon}$, which we are tempted to declare as unity. Let us keep this $\frac{\lambda}{\varepsilon}$ term separate from the order unity terms, even if we expect they may be of similar size. Then:

$$\overset{'}{R}_{ab} = \frac{\lambda}{\varepsilon^2}\Big[\exp\left(\frac{is}{\varepsilon}\right) i\, (\gamma^{c}_{ab}\, k_c - \gamma^{c}_{ac}\, k_b) + cc\Big] + \ldots$$

$$= \frac{\lambda}{\varepsilon^2}\Big[\exp\left(\frac{is}{\varepsilon}\right) (-\tfrac{1}{2})\, (\overset{o}{g}{}^{cd}\, h_{db}\, k_c\, k_a + \overset{o}{g}{}^{cd}\, h_{ac}\, k_d\, k_b$$

$$- h_{ab}\, \overset{o}{g}{}^{cd} k_c\, k_d - \overset{o}{g}{}^{cd}\, h_{cd}\, k_a\, k_b) + cc\Big] + \ldots \quad . \tag{5.15}$$

There is no $\frac{\lambda}{\varepsilon^2}$ term in T_{ab}, so the term displayed here must be zero. This is easily solved by k_a null with respect to the background:

$$\overset{o}{g}{}^{cd}\, k_c\, k_d = 0 \ , \tag{5.16}$$

and the "polarization tensor", h_{ab}, traceless and orthogonal to the propagation vector (with respect to $\overset{o}{g}_{ab}$):

$$\overset{o}{g}{}^{cd}\, h_{cd} = 0 \tag{5.17}$$

$$\overset{o}{g}{}^{cd}\, h_{ac}\, k_d = 0 \quad . \tag{5.18}$$

Then, by the same argument used before, k_a is also geodesic. These are the standard results of the geometric optics of gravitational waves.

Considering the electromagnetic case separately, the curl equation is solved by:

$$\overset{'}{A}_a = \exp\left(\frac{is}{\varepsilon}\right) \left(\frac{-i\varepsilon}{2}\omega_a\right) + cc + \ldots \tag{5.19}$$

$$\overset{'}{F}_{ab} = \exp\left(\frac{is}{\varepsilon}\right) \omega_{[a}\, k_{b]} + cc + \ldots \quad , \tag{5.20}$$

and the divergence equation is then:

$$0 = \exp\left(\frac{is}{\varepsilon}\right) \left(\frac{i}{\varepsilon}\ \overset{o}{g}{}^{ac}\, \overset{o}{g}{}^{bd}\, \omega_{[a}\, k_{b]}\, k_d\right) + cc \ + \ldots \quad , \tag{5.21}$$

implying:

$$\overset{o}{g}{}^{ab} k_a k_b = 0 \tag{5.22}$$

$$\overset{o}{g}{}^{ab} \omega_a k_b = 0 \quad . \tag{5.23}$$

The propagation vector is null, geodesic, and orthogonal to the polarization vector, with respect to the background.

At this point we should comment further on the relative size of λ and ε. The energy density of the electromagnetic wave is of order λ^2 (the leading terms in T_{ab}), ie, small, but the energy density of the gravitational wave is difficult to specify[69]. Note that the contracted curvature and the stress-energy-momentum are identified in Einstein's equations, yet a typical component of $R^a{}_{bcd}$ is of order $\frac{\lambda}{\varepsilon^2}$, which may diverge unless we make a stronger assumption than eq.(5.10), or rescale $h_{ab} \rightarrow \varepsilon^2 h_{ab}$, so that $R_{ab} \sim \lambda$ (In this context, the rescaling is a mathematical trick, and does not alter any results found so far). Rigorously, the relative acceleration of nearby test particles will diverge if $\frac{\lambda}{\varepsilon}$ is of order unity and $R_{ab} \sim \frac{\lambda}{\varepsilon^2}$. It appears we must be careful with the relative size of our two parameters to get well-behaved gravitational radiation.

-General Background

When there is a background F_{ab}, the mathematical complication increases. We apply high frequency assumptions for g'_{ab}, F'_{ab} to the general background perturbation equations, eq.(4.28)-(4.33) (starting with A'_a to automatically solve the curl equation):

$$A'_a = \exp\left(\frac{is}{\varepsilon}\right) \left(\frac{-i\varepsilon}{2} \omega_a\right) + cc + \ldots \tag{5.24}$$

$$F'_{ab} = \exp\left(\frac{is}{\varepsilon}\right) \omega_{[a} k_{b]} + cc + \ldots \tag{5.25}$$

$$g'_{ab} = \exp\left(\frac{is}{\varepsilon}\right) (- \varepsilon^2 h_{ab}) + cc + \ldots \quad . \tag{5.26}$$

We have chosen the particular forms here on energy density grounds. To make the equations look reasonable, we ask that the leading terms in energy

density(using the term loosely), which are simply the leading terms in Einstein's equation, be of order λ if ω_a and h_{ab} are of order unity. This is mainly a mathematical trick to avoid the problem of divergent $\overset{'}{R}_{ab}$. We can let $\lambda \to \lambda' \varepsilon^n$, and find equivalent equations for an expansion in λ', with all quantities rescaled by appropriate powers of ε. In the strong field high frequency case discussed earlier such rescaling is not possible. Much more important here is the relative size of $\overset{'}{g}_{ab}$ and $\overset{'}{F}_{ab}$. Physically, we can expect three distinguishable cases. If the source is primarily electromagnetic we may expect the resultant perturbation to have stronger (in energy) electromagnetic than gravitational part, and conversely, a neutral massive source might generate a mainly gravitational perturbation. Finally, a charged massive source likely generates the general case. These arguments are little more than extrapolations from simpler cases, but if we find freedom to adjust relative strengths, it will be tempting to say the physical cases are appearing. Since geometric optics always decouples the fields from the sources, the question cannot be settled in our framework. As far as working with eq.(5.24)-(5.26), let us consider that ω_a and h_{ab} may be as large as order unity, but they may be much smaller (so that the ratio of their magnitudes may vary).

Let us begin by assuming ω_a is order unity. We find, from eq.(4.28):

$$\overset{'}{T}_{ab} = \exp(\tfrac{is}{\varepsilon}) \left(\overset{o}{F}_a{}^c \omega_{[c}k_{b]} + \overset{o}{F}{}^c{}_b \omega_{[a}k_{c]} + \tfrac{1}{2} \overset{o}{g}_{ab} \overset{o}{F}{}^{ef} \omega_{[e}k_{f]}\right) + cc + \ldots \tag{5.27}$$

(again, indices of $\overset{o}{F}_{ab}$ have been raised with $\overset{o}{g}{}^{ab}$). Meanwhile:

$$\overset{'}{\Gamma}{}^a_{bc} = \tfrac{1}{2} \overset{o}{g}{}^{ad} \exp(\tfrac{is}{\varepsilon}) (-i\,\varepsilon) (h_{bd}k_c + h_{dc}k_b - h_{bc}k_d) + cc + \ldots \tag{5.28}$$

$$\overset{'}{R}_{ab} = \tfrac{1}{2} \overset{o}{g}{}^{cd} \exp(\tfrac{is}{\varepsilon}) (h_{ad}k_b k_c + h_{db}k_a k_c - h_{ab}k_d k_c - h_{dc}k_a k_b) + cc + \ldots . \tag{5.29}$$

For the electromagnetic divergence equation (eq(4.32)):

$$\overset{\prime}{F}{}^{ab} = \overset{o}{g}{}^{ac}\,\overset{o}{g}{}^{bd}\,\overset{\prime}{F}_{cd} + \dots \tag{5.30}$$

$$\overset{\prime}{F}{}^{ab}{}_{;b} = \left(\overset{o}{g}{}^{ac}\,\overset{o}{g}{}^{bd}\,\overset{\prime}{F}_{cd} \right)_{\underset{o}{;b}} + \dots \tag{5.31}$$

$$= \overset{o}{g}{}^{ac}\,\overset{o}{g}{}^{bd} \exp\left(\tfrac{is}{\varepsilon}\right)\left(\tfrac{i}{\varepsilon}\right)\left(\omega_c k_d k_b - \omega_d k_c k_b \right) + cc + \dots$$

$$= 0 \quad .$$

This is the same form seen in the case of no electromagnetic background:

$$\overset{o}{g}{}^{bd}\, k_b\, k_d = 0 \tag{5.32}$$

$$\overset{o}{g}{}^{bd}\, k_b\, \omega_d = 0 \quad . \tag{5.33}$$

The leading terms of Einstein's equation then imply:

$$\tfrac{1}{2}\overset{o}{g}{}^{cd}\left(h_{ad}k_bk_c + h_{db}k_ak_c - h_{dc}k_ak_b \right) = \overset{o}{F}{}_a{}^{c}\omega_{[c}k_{b]} + \overset{o}{F}{}^{c}{}_{b}\omega_{[a}k_{c]} + \overset{o}{g}_{ab}\overset{o}{F}{}^{ef}\omega_{[e}k_{f]} \quad . \tag{5.34}$$

It is possible that the leading terms of $\overset{\prime}{T}_{ab}$ (eq(5.27)) are zero, but this requires special coupling of $\overset{\prime}{F}_{ab}$ to $\overset{o}{F}_{ab}$, eg:

$$\overset{o}{g}{}^{bc}\,\overset{o}{F}_{ab}\,k_c = 0 \quad . \tag{5.35}$$

This seems restrictive. For instance, no radially travelling high frequency $\overset{\prime}{F}_{ab}$ in the Reissner-Nordstrom geometry can satisfy eq(5.35). Since it appears that nonzero eq(5.27) is the general case, we find that h_{ab} must be order unity, and that it must satisfy a rather complicated equation linking it to the electromagnetic quantities. Since the electromagnetic perturbation travels as freely as in the no-electromagnetic-background case, but the gravitational perturbation cannot be set to zero because it is coupled to $\overset{\prime}{F}_{ab}$, an appealing interpretation is that this configuration is primarily an electromagnetic disturbance, generating a sympathetic gravitational disturbance through interaction with the background.

Now let us find the minimal $\overset{\prime}{F}_{ab}$ required when there is a gravita-

tional perturbation, due to interaction with $\overset{o}{F}_{ab}$. We take eq(5.26) with h_{ab} of order unity, so that $\overset{'}{R}_{ab}$ is of order $\lambda\varepsilon^0$. Obviously, if ω_a is any smaller than order unity, the leading terms in Einstein's equation are purely gravitational, signalling a gravitational wave not interacting with any electromagnetic field, and we find:

$$\overset{o}{g}{}^{cd}\, h_{cd} = 0 \tag{5.36}$$

$$\overset{o}{g}{}^{cd}\, h_{ac}\, k_d = 0 \tag{5.37}$$

$$\overset{o}{g}{}^{cd}\, k_c\, k_d = 0. \tag{5.38}$$

In the electromagnetic equations, $\overset{'}{F}{}^{ab}$ contains terms like:

$$\overset{o}{g}{}^{ac}\, \overset{,}{g}{}^{bd}\, \overset{o}{F}_{cd} = -\overset{o}{g}{}^{ac}\, \overset{o}{g}{}^{be}\, \overset{o}{g}{}^{df}\, \overset{,}{g}_{ef}\, \overset{o}{F}_{cd}\ , \tag{5.39}$$

so $\overset{'}{F}{}^{ab}$ is of order ε^2 even if there is no $\overset{'}{F}_{ab}$. Then $\overset{'}{F}{}^{ab}{}_{;b}$ has several terms in order ε . If $\overset{'}{F}_{ab}$ is zero:

$$\begin{aligned} \overset{'}{F}{}^{ab} &= \overset{o}{g}{}^{ac}\, \overset{,}{g}{}^{bd}\, \overset{o}{F}_{cd} + \overset{,}{g}{}^{ac}\, \overset{o}{g}{}^{bd}\, \overset{o}{F}_{cd} + \ldots \\ &= \overset{o}{F}{}^{a}{}_{d}(-\overset{o}{g}{}^{be}\, \overset{o}{g}{}^{df}\, \overset{,}{g}_{ef}) + \overset{o}{F}{}_{c}{}^{b}(-\overset{o}{g}{}^{ae}\, \overset{o}{g}{}^{cf}\, \overset{,}{g}_{ef}) + \ldots \\ &= \exp(\tfrac{is}{\varepsilon})\varepsilon^2(\overset{o}{F}{}^{af}\, \overset{o}{g}{}^{be}\, h_{ef} + \overset{o}{F}{}^{fb}\, \overset{o}{g}{}^{ae}\, h_{ef}) + cc + \ldots\ , \end{aligned} \tag{5.40}$$

and eq(4.32) becomes:

$$0 = \overset{'}{F}{}^{ab}{}_{,b} + \overset{o}{F}{}^{ac}\, \overset{'}{\Gamma}{}^{b}{}_{cb} + \overset{o}{F}{}^{cb}\, \overset{'}{\Gamma}{}^{a}{}_{cb} + \ldots\ , \tag{5.41}$$

which simplifies using eq(5.36)-(5.38) to:

$$0 = \overset{o}{g}{}^{ae}\, h_{ef}\, \overset{o}{F}{}^{bf}\, k_b \tag{5.42}$$

(leading term). This is an extra restriction on the gravitational perturbation (which is undesirable) that can be avoided if $\overset{'}{F}_{ab}$ is of order ε^2:

$$\overset{'}{F}_{ab} = \exp(\tfrac{is}{\varepsilon})\ \varepsilon^2 \omega_{[a}\, k_{b]} + cc + \ldots \tag{5.43}$$

$$\overset{'}{F}{}^{ab} = \exp(\tfrac{is}{\varepsilon})\varepsilon^2(\overset{o}{g}{}^{ac}\, \overset{o}{g}{}^{bd}\, \omega_{[c}k_{d]} + h_{cd}(\overset{o}{g}{}^{bc}\, \overset{o}{F}{}^{ad} + \overset{o}{g}{}^{ac}\, \overset{o}{F}{}^{db})) + cc + \ldots\ . \tag{5.44}$$

Instead of eq(5.42) we find:

$$0 = \tfrac{1}{2}\, \overset{o}{g}{}^{ac}\, \overset{o}{g}{}^{bd}\, \omega_d\, k_c\, k_b - \overset{o}{g}{}^{ac}\, h_{cd}\, \overset{o}{F}{}^{bd}\, k_b \ , \qquad (5.45)$$

indicating that F'_{ab} is coupled to the background. From the same kind of argument as before, this appears to be a source-generated gravitational perturbation, producing an electromagnetic disturbance through coupling to the background.

Between the two cases discussed so far lies F'_{ab} of order ε, h_{ab} of order unity. On the basis of our physical arguments, this should mean both types of radiation are driven by active sources, in which case the ε appearing in the energy density of F'_{ab} may seem strange. It simply means that the relative amplitudes of the two disturbances for which geometric optics can identify both as source-driven (if this is the proper interpretation of this case) are frequency dependent. Inspection of the field equations for this case reveals that, in the leading order, the two radiation fields are independent of each other, and propagate without explicit regard for the background electromagnetic field (ie, along null geodesics as found in the case of a purely gravitational background).

In the case of independent leading orders (the last case discussed), we can again use independent s_1, s_2, and ε_1, ε_2, for the fields. In the other two cases, we have assumed that the sympathetic field (generated by the interaction of the applied field with the background) is of the same frequency as the applied field, and have written the equations for the sympathetic field with that assumption. We have not solved those equations because they involve the background and must be considered on a case-by-case basis. If in some particular case the leading order equations in the sympathetic field do not have reasonable solution, then

the frequency assumption, and the use of geometric optics, should be reconsidered for that particular case.

In considering all three cases we see that a high frequency perturbation which is applied to the background, and not generated by the interaction of another field with the background, travels along null geodesics of the background, in first approximation.

Geometric Optics In BMB

- Minkowski Background

We noted that the EMT equations for the first order of a perturbation of Minkowski space have plane wave solutions, without any restriction on frequency. Mimicing a standard method of solution in EMT for the analogous case in BMB, we ask for harmonic coordinates:

$$g^{ab}\Gamma^{c}_{ab} = 0 \quad , \tag{5.46}$$

at least in first order, where this implies:

$$\eta^{ab}\, g'_{(bc),a} = \tfrac{1}{2}\eta^{ab}\, g'_{(ab),c} \quad . \tag{5.47}$$

Then, note that if the first order BMB fields satisfy eq(5.47) and (using eq(2.11),(2.17) for the generalized electromagnetic field and potential):

$$F'_{ab} = A'_{a,b} - A'_{b,a} \tag{5.48}$$

$$\Box^2 A'_a = 0 \tag{5.49}$$

$$\eta^{ab} A'_{a,b} = C \tag{5.50}$$

$$\Box^2 g'_{(ab)} = 0 \tag{5.51}$$

(C is a constant) then they satisfy eq(4.51)-(4.53). Eq(5.47)-(5.51) are (setting C to zero) just the usual EMT equations in Lorentz gauge for these circumstances, and they have the standard plane wave solutions:

$$A'_a(x) = \omega_a \exp(ik_c x^c) + cc \tag{5.52}$$

$$\overset{\prime}{g}_{ab}(x) = h_{ab} \exp(ip_c x^c) + cc \tag{5.53}$$

where:

$$\eta^{ab} k_a k_b = 0 \tag{5.54}$$

$$\eta^{ab} p_a p_b = 0 \tag{5.55}$$

$$\eta^{ab} k_a \omega_b = 0 \tag{5.56}$$

$$\eta^{ab} p_a h_{bc} = \tfrac{1}{2} \eta^{ab} h_{ab} p_c \tag{5.57}$$

and k_a, p_a, ω_b, h_{bc} are constants. k_a and p_a define straight lines (of propagation), which are the geodesics of Minkowski space (to first order $k_{a;b}$ is itself zero).

In chapter 4 we made some comments about the second order of perturbation. In particular, if the first order is purely electromagnetic (ie, $\overset{\prime}{g}_{(ab)}$ is zero), then the second order can be purely gravitational (ie, $\overset{\prime\prime}{g}_{[ab]}$ zero), satisfying EMT's Einstein equation with correction terms of order k^2 (This k is the new coupling constant). In the case of first order electromagnetic plane wave, eq(5.52),(5.54),(5.56), the order k^2 terms in the second order gravitational equation are all zero (see Appendix), so that only Einstein's equation is left:

$$\overset{\prime\prime}{R}_{ab(emt)} = 8\pi \overset{\prime\prime}{T}_{ab(emt)} \quad . \tag{5.58}$$

Thus, for plane electromagnetic waves perturbing Minkowski background, EMT and BMB are not distinguishable to second order in amplitude.

- Gravitational Background

We noted in chapter 4 that in a purely gravitational background the BMB $\overset{\prime}{g}_{(ab)}$ perturbation is independent of $\overset{\prime}{g}_{[ab]}$, and satisfies equations identical to those of the analogous general relativity case. The results for radiation in this case are well known[61], and have already been discussed. For the electromagnetic field, $\overset{\prime}{g}_{[ab]}$:

$$\overset{\prime}{g}_{[ab]} = \exp(\tfrac{is}{\varepsilon})\,(-\varepsilon^2\, h_{[ab]}) + cc + \ldots, \tag{5.59}$$

we must reconsider our arguments about relative amplitudes. In EMT we argued that in high frequency $\overset{\prime}{g}_{(ab)}$ must be no larger than $O(\varepsilon^2)$ and $\overset{\prime}{F}_{ab}$ no larger than $O(1)$ so that energy as it appears in Einstein's equation is $O(\lambda)$. Here, $\overset{\prime}{g}_{(ab)}$ and $\overset{\prime}{g}_{[ab]}$ are on an entirely equal footing, and if we still ask that typical curvature components have amplitude λ, then $\overset{\prime}{g}_{[ab]}$ must be of order ε^2. This appears to be a decided change from EMT, but it remains to be seen whether it generates new properties.

Let us now examine the leading order terms in the field equations:

$$\overset{\prime}{\Gamma}{}^{a}_{bc} = \exp\,(\tfrac{is}{\varepsilon})\,(-\tfrac{i\varepsilon}{2})\,\overset{o}{g}{}^{ad}(h_{[bd]}k_c + h_{[dc]}k_b + h_{[bc]}k_d) + cc + \ldots\,, \tag{5.60}$$

but this must be traceless, implying:

$$\overset{o}{g}{}^{cd}\, k_c\, h_{[bd]} = 0\ . \tag{5.61}$$

Now:

$$\begin{aligned} \overset{\prime}{R}_{[ab]} &= \overset{\prime}{\Gamma}{}^{c}_{[ab],c} + \ldots \\ &= \exp\,(\tfrac{is}{\varepsilon})\,\tfrac{1}{2}\,\overset{o}{g}{}^{cd}(h_{[ad]}k_b + h_{[db]}k_a + h_{[ab]}k_d)\,k_c + cc + \ldots \\ &= \exp\,(\tfrac{is}{\varepsilon})\,\tfrac{1}{2}\,\overset{o}{g}{}^{cd}\,k_c k_d h_{[ab]} + cc + \ldots \end{aligned} \tag{5.62}$$

while $\overset{\prime}{I}_{ab}$ is of order ε^2. Therefore the field equation on the curvature is:

$$\overset{o}{g}{}^{cd}\,k_c k_d h_{[ab,e]} = 0\ , \tag{5.63}$$

which is solved by :

$$\overset{o}{g}{}^{cd}\,k_c k_d = 0 \tag{5.64}$$

or

$$h_{[ab,e]} = 0\ . \tag{5.65}$$

If eq. (5. 65) holds then there exists some vector v_a, not proport-

ional to k_a, such that:

$$h_{[ab]} = v_{[a}k_{b]} \ , \tag{5.66}$$

but we must have:

$$\begin{aligned} 0 &= \overset{o}{g}{}^{ab} k_a h_{[cb]} \\ &= \tfrac{1}{2} (\overset{o}{g}{}^{ab} k_a v_c k_b - \overset{o}{g}{}^{ab} k_a v_b k_c), \end{aligned} \tag{5.67}$$

which can only be solved by;

$$\overset{o}{g}{}^{ab} k_a k_b = 0 \tag{5.68}$$

and:

$$\overset{o}{g}{}^{ab} k_a v_b = 0 \ . \tag{5.69}$$

The conclusion that can now be drawn is that to satisfy eq.(5.63) k_a must be null with respect to the background. Then by working through the same argument used in EMT, we find:

$$\overset{o}{g}{}^{ab} k_a k_{c\overset{;}{o}b} = 0 \ , \tag{5.70}$$

that is, the paths generated by k_a are geodesics of the background, to first order.

Notice in the above argument that we have not yet seen anything to do with the noncompatible BMB connection. Since the background is purely gravitational, it is ruled by general relativity, where the connection is compatible with the metric. To first order, only the back - ground connection appears in the geodesic equation. To gain information on whether radiation does follow the BMB noncompatible connection we must either solve higher orders here (which would require detailed knowledge of the background) or leading order in the general-background case, which we will attempt to do.

- General Background

Using the high frequency form:

$$g'_{ab} = \exp\left(\frac{is}{\varepsilon}\right) (- \varepsilon^2 h_{ab}) + cc + \ldots \tag{5.71}$$

(with both symmetric and skew parts) in eq. (4.65)-(4.70), we find for the connection (due to the high frequency):

$$\overset{o}{g}_{(cd)} \overset{\prime}{\Gamma}{}^{d}_{(ab)} = \tfrac{1}{2} (\overset{\prime}{g}_{(cb),a} + \overset{\prime}{g}_{(ac),b} - \overset{\prime}{g}_{(ab),c}) - \overset{o}{g}_{[db]} \overset{\prime}{\Gamma}{}^{d}_{[ca]} - \overset{o}{g}_{[ad]} \overset{\prime}{\Gamma}{}^{d}_{[bc]} \tag{5.72}$$

$$\overset{o}{g}_{(cd)} \overset{\prime}{\Gamma}{}^{d}_{[ab]} = \tfrac{1}{2} (\overset{\prime}{g}_{[cb],a} + \overset{\prime}{g}_{[ac],b} + \overset{\prime}{g}_{[ab],c}) - \overset{o}{g}_{[db]} \overset{\prime}{\Gamma}{}^{d}_{(ca)} - \overset{o}{g}_{[ad]} \overset{\prime}{\Gamma}{}^{d}_{(bc)} . \tag{5.73}$$

To get equations for $\overset{\prime}{\Gamma}{}^{a}_{bc}$ we must contract these with $\overset{o}{\tau}{}^{ab}$, the inverse of $\overset{o}{g}_{(ab)}$, which does exist for a well behaved background metric. However, doing so resembles "raising indices with the symmetric part of the metric", contrary to the spirit of chapter 3. Unfortunately, we have no choice if we wish to go on. It does at least seem consistent to assume:

$$\overset{\prime}{\Gamma}{}^{a}_{bc} = \exp(\tfrac{is}{\varepsilon}) \; (- i\varepsilon \gamma^{a}_{bc}) + cc + \dots \quad . \tag{5.74}$$

Then:

$$\gamma^{c}_{(ab)} = \overset{o}{\tau}{}^{ce} \left[\tfrac{1}{2}(h_{(eb)}k_a + h_{(ae)}k_b - h_{(ab)}k_e) - \overset{o}{g}_{[db]} \gamma^{d}_{[ea]} - \overset{o}{g}_{[ad]} \gamma^{d}_{[be]} \right] \tag{5.75}$$

$$\gamma^{c}_{[ab]} = \overset{o}{\tau}{}^{ce} \left[\tfrac{1}{2}(h_{[eb]}k_a + h_{[ae]}k_b + h_{[ab]}k_e) - \overset{o}{g}_{[db]} \gamma^{d}_{(ea)} - \overset{o}{g}_{[ad]} \gamma^{d}_{(be)} \right] . \tag{5.76}$$

These forms do not allow us to immediately express γ^{a}_{bc} in terms of h_{ab} and k_c because of the interaction of $\gamma^{a}_{(bc)}$ with $\gamma^{a}_{[bc]}$ through $\overset{o}{g}_{[ab]}$.

Proceeding on to the other equations:

$$\begin{aligned} 0 &= \gamma^{c}_{[ac]} \\ &= \overset{o}{\tau}{}^{ce} h_{[ae]}k_c - \overset{o}{\tau}{}^{ce} \overset{o}{g}_{[dc]} \gamma^{d}_{(ea)} - \overset{o}{\tau}{}^{ce} \overset{o}{g}_{[ad]} \gamma^{d}_{(ce)} \quad . \end{aligned} \tag{5.77}$$

$\overset{\prime}{I}_{ab}$, involving no derivatives, is much smaller than the leading curvature terms, and can be ignored. Thus the curvature equations (leading terms) are:

$$0 = \gamma^{c}_{(ab)}k_c - \tfrac{1}{2}(\gamma^{c}_{(ac)}k_b + \gamma^{c}_{(bc)}k_a) \qquad (5.78)$$

$$2\dot{A}_{[a,b]} = \exp(\tfrac{is}{\varepsilon})\ k_c\ \gamma^{c}_{[ab]} + cc + \ldots\ . \qquad (5.79)$$

With the reasonable assumption:

$$\dot{A}_b = \exp(\tfrac{is}{\varepsilon})\ (-i\,\varepsilon\, a_b) + cc + \ldots \qquad (5.80)$$

we find:

$$2\, a_{[b}k_{c]} = k_d\,\gamma^{d}_{[bc]}\ , \qquad (5.81)$$

which is solved by :

$$a_b = k_d\ p^d\ q_b \qquad (5.82)$$

$$\gamma^{d}_{[bc]} = 2\ p^d\ q_{[b}k_{c]}\ . \qquad (5.83)$$

Then the trace equation says k_a and p^a are orthogonal, which means:

$$k_c\,\gamma^{c}_{[ab]} = 0 \qquad (5.84)$$

and $\dot{R}_{[ab]}$'s leading order is itself zero. Eq.(5.84) is probably the strongest thing that can be concluded (rather than any of the intermediate steps). Using eq.(5.76):

$$\begin{aligned} 0 &= k_c\,\gamma^{c}_{[ab]} \\ &= \tfrac{1}{2}\,(h_{[eb]}\,\overset{o}{\tau}{}^{ce}\,k_c k_a + h_{[ae]}\,\overset{o}{\tau}{}^{ce}\,k_c k_b \\ &\quad + h_{[ab]}\,\overset{o}{\tau}{}^{ce}\,k_c k_e) - \overset{o}{g}_{[db]}\,\gamma^{d}_{(ea)}\,\overset{o}{\tau}{}^{ce}\,k_c \\ &\quad - \overset{o}{g}_{[ad]}\,\gamma^{d}_{(be)}\,\overset{o}{\tau}{}^{ce}\,k_c\ . \end{aligned} \qquad (5.85)$$

At this point, further progress becomes difficult. There is no obvious solution to eq.(5.85), but we are hoping for simple results, perhaps:

$$\overset{o}{\tau}{}^{ce}\,k_c k_e = 0 \qquad (5.86)$$

$$\overset{o}{\tau}{}^{ce}\,h_{[ae]}k_c = 0 \qquad (5.87)$$

and

$$\overset{o}{\tau}{}^{ce}\ \gamma^{d}_{(ea)}k_c = 0\ . \qquad (5.88)$$

These three together will solve eq.(5.85), but eq.(5.86) in eq.(5.77) implies a coupling of first order to the background and eq.(5.88) and

(5.75) together imply a different equation coupling first order to background. Such (extra) couplings make the simple results we are looking for unlikely.

The same simple-minded techniques can be applied to the equation in symmetric curvature. To satisfy eq.(5.78) we might assume:

$$\gamma^{c}_{(ac)} = 0 \tag{5.89}$$

$$k_c \, \gamma^{c}_{(ab)} = 0 \tag{5.90}$$

from which we find:

$$\begin{aligned} 0 &= \gamma^{c}_{(ac)} \\ &= \tfrac{1}{2} \, \overset{o}{\tau}{}^{ce} h_{(ec)} k_a - \overset{o}{\tau}{}^{ce} \overset{o}{g}_{[dc]} \gamma^{d}_{[ea]} \end{aligned} \tag{5.91}$$

$$\begin{aligned} 0 &= k_c \, \gamma^{c}_{(ab)} \\ &= \tfrac{1}{2} \left(h_{(eb)} \, \overset{o}{\tau}{}^{ce} k_c k_a + h_{(ae)} \, \overset{o}{\tau}{}^{ce} k_c k_b \right. \end{aligned} \tag{5.92}$$

$$\left. - h_{(ab)} \, \overset{o}{\tau}{}^{ce} k_c k_e \right) - \overset{o}{g}_{[db]} \gamma^{d}_{[ea]} \overset{o}{\tau}{}^{ce} k_c - \overset{o}{g}_{[ad]} \gamma^{d}_{[be]} \overset{o}{\tau}{}^{ce} k_c \, . \tag{5.92}$$

In the same way as before, trying to assume simple forms here results in extra coupling of background to foreground. We fall into calculational confusion.

Having been frustrated in this approach, we can try changing the relative size of $g'_{(ab)}$ and $g'_{[ab]}$ (in analogy with the EMT section).

If we take:

$$g'_{[ab]} = \exp\left(\frac{is}{\varepsilon}\right) \left(-\varepsilon^2 h_{[ab]}\right) + cc + \ldots \tag{5.93}$$

$$g'_{(ab)} = 0\,(\varepsilon^3) \tag{5.94}$$

then:

$$\gamma^{c}_{(ab)} = - \overset{o}{\tau}{}^{ce} \overset{o}{g}_{[db]} \gamma^{d}_{[ea]} - \overset{o}{\tau}{}^{ce} \overset{o}{g}_{[ad]} \gamma^{d}_{[be]} \tag{5.95}$$

$$\gamma^{c}_{[ab]} = \overset{o}{\tau}{}^{cd} \left[\tfrac{1}{2} \left(h_{[db]} k_a + h_{[ad]} k_b + h_{[ab]} k_d \right) \right. \tag{5.96}$$

$$- \overset{o}{\tau}^{ef}\Big(\overset{o}{g}_{[fb]}\overset{o}{g}_{[jd]}\gamma^{j}_{[ea]} + \overset{o}{g}_{[fb]}\overset{o}{g}_{[aj]}\gamma^{j}_{[de]} + \overset{o}{g}_{[af]}\overset{o}{g}_{[jd]}\gamma^{j}_{[eb]} + \overset{o}{g}_{[af]}\overset{o}{g}_{[bj]}\gamma^{j}_{[de]} \Big)\Big] .$$

Here, instead of $\gamma^{c}_{[ab]}$ expressed in terms of $h_{[ab]}$ and k_a (which we expected), we find a complicated expression in $\gamma^{c}_{[ab]}$ and the background electromagnetic field. Without an easily solved relation between g'_{ab} and γ^{c}_{ab}, it becomes difficult to draw any conclusions from the rest of the field equations. The author has had little success in assuming simple forms (in analogy with EMT) in this case or in the case of $g'_{(ab)}$ larger than $g'_{[ab]}$.

In summarizing the failure of this approach we can make several remarks. Whereas in the earlier cases, there seemed to be a fair analogy between EMT and BMB radiation, there is no similarity apparent here. The elementary techniques we have used seemed to work with ridiculous ease in EMT by contrast with this section. We found that when there is a leading order $g'_{[ab]}$ it does not generate any generalized electromagnetic potential at the order expected, so the leading order $g'_{[ab]}$ disappears in the EMT limit, probably placing this disturbance in the "special" category (see eq(2.96), (2.97)) discussed in chapter 2. Since we have as much trouble with $g'_{(ab)}$ as with $g'_{[ab]}$, this is little consolation. Lastly, note the persistent appearance of $\overset{o}{\tau}^{ab}$, versus the absence of $\overset{o}{g}^{ab}$ or even $\overset{o}{g}^{(ab)}$. It appears from the work done here that $\overset{o}{\tau}^{ab}$ is more important for solving the equations than either of its competitors. Whether it is more important physically is yet to be seen.

Note further that if we had found k_a null with repect to $\overset{o}{\tau}^{ab}$, then k_a would be geodesic with respect to $\overset{o}{\Gamma}{}^{c}_{(ab)lc}$:

$$\overset{o}{\tau}^{ab} k_a k_{c;b \atop olc} = 0 \tag{5.97}$$

and would follow a path which is an extremal of arc length (the proof is exactly that used in EMT). On the other hand, if k_a is null with respect to $\overset{o}{g}{}^{ab}$:

$$0 = \overset{o}{g}{}^{ab} k_a k_b$$
$$= \overset{o}{g}{}^{(ab)} k_a k_b , \qquad (5.98)$$

it is null with respect to $\overset{o}{g}{}^{(ab)}$. Corresponding to $g^{(ab)}$ there is a symmetric connection (call it $\Gamma^a_{(bc)1c2}$) defined by

$$g^{(ab)}{}_{;c \atop 1c2} = 0 , \qquad (5.99)$$

and this time k_a follows geodesics of $\Gamma^a_{(bc)1c2}$:

$$\overset{o}{g}{}^{(ab)} k_a k_{c;b \atop o1c2} = 0 . \qquad (5.100)$$

By the same type of calculation done to show that Γ^a_{bc} and $\Gamma^a_{(bc)1c}$ are not related by a projective transformation[54], we can see that $\Gamma^a_{(bc)1c2}$ generates geodesics different than those of Γ^a_{bc} and of $\Gamma^a_{(bc)1c}$. Thus the geometric optics formulation we have developed here discriminates against the most expected case: if high frequency radiation travels along null geodesics of Γ^a_{bc} " in the first approximation", then the equations here will be of complicated form. Possibly the standard analysis could be modified to single out null geodesics of Γ^a_{bc} instead of $\Gamma^a_{(bc)1c}$, which the work here seems to be doing (but the form of such modification is not obvious to the author).

- Other Cases

The case of high frequency without weak fields was not presented earlier because it contains difficulties. The obvious form to assume is:

$$g_{[ab]} = \exp\left(\frac{is}{\varepsilon}\right) h_{[ab]} + cc \qquad (5.101)$$

as the generalized electromagnetic field in some region where the BMB equations are satisfied exactly. The field equation relating g_{ab} and

Γ^a_{bc} then implies:

$$\Gamma^c_{(ab)} = \tau^{ce}\Big[\tfrac{1}{2}\left(g_{(eb),a} + g_{(ae),b} - g_{(ab),e}\right) - \left(\exp\left(\tfrac{is}{\varepsilon}\right)\left(h_{[db]}\Gamma^d_{[ea]} + h_{[ad]}\Gamma^d_{[be]}\right) + cc\right)\Big] \tag{5.102}$$

(where τ^{ab} is the inverse of $g_{(ab)}$)

$$\Gamma^c_{[ab]} = \tau^{ce}\Big[\tfrac{1}{2}\,\tfrac{i}{\varepsilon}\exp\left(\tfrac{is}{\varepsilon}\right)\left(h_{[eb]}k_a + h_{[ae]}k_b + h_{[ab]}k_e\right) - \exp\left(\tfrac{is}{\varepsilon}\right)\left(h_{[db]}\Gamma^d_{(ea)} + h_{[ad]}\Gamma^d_{(be)}\right) + cc\Big] + 0(1) . \tag{5.103}$$

If we substitute eq.(5.102) into (5.103), or vice versa , we find an expression with $\Gamma^a_{[bc]}$ (or $\Gamma^a_{(bc)}$) related to itself in a complicated manner. Now, we would like to assume as little as possible about $g_{(ab)}$. Since the curvature $R^a{}_{bcd}$ involves second derivatives of g_{ab}, it appears that curvature is of order $\frac{1}{\varepsilon^2}$, due to $g_{[ab]}$ (we will discuss the reasonableness of this later). If we ask that $R^a{}_{bcd}$ be no larger than this, then $g_{(ab),c}$ should be no larger than $0(\frac{1}{\varepsilon})$. If it is of this maximum size, then $g_{(ab),c}$ contributes directly to $\Gamma^a_{[bc]}$. Worse than this are terms which relate $\Gamma^a_{[bc]}$ to itself through $h_{[ab]}$. In these we see factors like $(\exp(\frac{is}{\varepsilon}))^2$. These indicate intermodulation, where waves interfere to produce new frequencies, including higher and lower frequencies than those of the original waves. Our geometric optics forms, however, have assumed a narrow band of frequencies, and this is not likely to be the case when there is intermodulation. This case seems difficult indeed.

Possibly one of the basic problems with this formulation of non-weak high frequency radiation is the size of the curvature. Under normal circumstances, the curvature must be finite at x for x to be considered part of physical space. In the context of high frequency, Isaacson[70] allows $R^a{}_{bcd}$ to be $0(\frac{1}{\varepsilon})$ in a particular situation in general relativity, and then argues that this will not cause unreasonable results in that

particular situation. In the EMT non-weak field case, T_{ab} is order unity, so the electromagnetic field only generates order unity curvature. The mathematics of that case work nicely only if $R^a{}_{bcd}$ is smaller than $O(\frac{1}{\varepsilon^2})$: we assumed that $f^{ab}{}_{;c}$ is order unity:

$$f^{ab}{}_{;c} = f^{ab}{}_{,c} + f^{ad}\Gamma^b_{dc} + f^{db}\Gamma^a_{dc}, \tag{5.104}$$

and this is easily so providing Γ^a_{bc} is order unity (ie, $R^a{}_{bcd} \sim (\frac{1}{\varepsilon})$ or less). Thus the BMB work with $R^a{}_{bcd}$ of order $\frac{1}{\varepsilon^2}$ may not be physically meaningful. However, no alternate to eq.(5.101) immediately presents itself for the investigation of high frequency without weak fields.

So far we have not been able to actually solve for paths of motion when the connection contains obvious unified field modifications. As a last attempt consider multiplying the field of eq.(5.101) by ε. The result will be a one-parameter weak high frequency field, which is not as well motivated as the two parameter fields but is easier to manipulate mathematically. Curvature will still be divergent, but now only as $\frac{1}{\varepsilon}$. Considered as a perturbation to a background without electromagnetic field (and allowing a possible gravitational perturbation of similar form):

$$g_{[ab]} = \varepsilon \exp(\tfrac{is}{\varepsilon})\, h_{[ab]} + cc + \dots \tag{5.105}$$

$$g_{(ab)} = \overset{o}{g}_{(ab)} + \varepsilon \exp(\tfrac{is}{\varepsilon})\, h_{(ab)} + cc + \dots \quad . \tag{5.106}$$

$\overset{o}{g}_{(ab)}$ is a solution in general relativity.[71] This case is in a weak sense analogous to the one considered by Isaacson [70], and, in a different sense, can be obtained from the earlier two parameter perturbation on a gravitational background by rescaling so that the curvature is of order $\frac{\lambda}{\varepsilon^2}$ and setting λ equal to ε.

$$g^{ab} = \overset{o}{g}{}^{(ab)} + \varepsilon \exp(\tfrac{is}{\varepsilon})(h^{(ab)} + h^{[ab]}) + cc + \dots \tag{5.107}$$

$$\Gamma^c_{(ab)} = \tfrac{1}{2}\overset{o}{g}{}^{(ce)}(\overset{o}{g}_{(eb),a} + \overset{o}{g}_{(ae),b} - \overset{o}{g}_{(ab),e})$$
$$+ \frac{i}{2}\exp(\tfrac{is}{\varepsilon})\,\overset{o}{g}{}^{(ce)}(h_{(eb)}k_a + h_{(ae)}k_b - h_{(ab)}k_e) + cc + O(\varepsilon) \tag{5.108}$$

$$\Gamma^{c}_{[ab]} = \exp(\tfrac{is}{\varepsilon})\,\tfrac{i}{2}\,\overset{o}{g}{}^{(ce)}\,(h_{[eb]}k_a + h_{[ae]}k_b + h_{[ab]}k_e) + cc + O(\varepsilon)\ . \tag{5.109}$$

Note the symmetric part of the connection will have order unity high frequency terms if $h_{(ab)}$ is nonzero, while the torsion is order unity, and high frequency. Thus, $\Gamma^{c}_{(ab)lc}$, which is to order unity eq.(5.108), differs from the field equation connection in order unity. Working through the field equations in the same manner as was done in the two-parameter gravitational background case:

$$\overset{o}{g}{}^{ce}\,h_{[ac]}k_e = 0 \tag{5.110}$$

$$R_{[ab]} = \frac{-1}{\varepsilon}\exp(\tfrac{is}{\varepsilon})\,\overset{o}{g}{}^{(ce)}h_{[ab]}k_e k_c + cc + \ldots\ , \tag{5.111}$$

and eventually:

$$\overset{o}{g}{}^{(ce)}k_c k_e = 0\ . \tag{5.112}$$

Furthermore, the equation on $R_{(ab)}$ is strictly in $h_{(ab)}$, so $h_{(ab)}$ and $h_{[ab]}$ are independent in leading order, we can set $h_{(ab)}$ to zero if we so desire, and we can attach independent $s^{(i)}$ and $k^{(i)}_a$ to the two independent perturbations. Anyway, the electromagnetic perturbation moves on null geodesics of the connection compatible with the slowly varying background metric:

$$\overset{o}{\Gamma}{}^{c}_{(ab)lc} = \tfrac{1}{2}g^{o(ce)}(\overset{o}{g}_{(eb),a} + \overset{o}{g}_{(ae),b} - \overset{o}{g}_{(ab),e})\ . \tag{5.113}$$

At first glance, it would seem that in this case the Levi-Civita connection is definitely singled out, and the field equation connection, being the same size, is definitely avoided. Unfortunately, there is another consideration. If we consider $h_{[ab]}$ zero, $h_{(ab)}$ nonzero, we have a gravitational perturbation in general relativity, in fact, the leading order of the case considered by Isaacson. In it we find the connection, simply eq.(5.108), has order unity high and low frequency

parts, but the radiation travels on null geodesics of the low frequency part alone, eq.(5.113). Thus we once again cannot conclude anything about the relative validity of Γ^a_{bc} vs $\Gamma^a_{(bc)lc}$, since it may be that the low frequency part of the connection (whichever connection is correct) is singled out in this case.

Summary

In this chapter we have applied the geometric optics approximation, in a form extended slightly beyond that of introductory relativity texts, to EMT and BMB. With it, radiation in EMT was found to behave in the intuitively (even naively) expected manner in all interesting cases tested. In BMB, while we did find nice results in simple circumstances, we were never able to solve any case that would have truly discriminated between $\Gamma^a_{(bc)lc}$ and Γ^a_{bc}, and thus begin to resolve some of the questions raised in chapter 3. Possibly we could evaluate the non-weak-field and general background cases using Tonnelat's[46] method of finding Γ^c_{ab}, but this is beyond the scope of this thesis. We could try to test the general background case using the exact, spherically symmetric, static solution as a particular background. While this might help rule out some of the many possible radiation trajectories, it would be difficult to identify the general rule from one such example. Lastly, geometric optics is not the only way to study radiation. The study of characteristics (as mathematical idealizations of shock wave fronts) has proven fruitful in general relativity[64], but is once again beyond the scope of this thesis.

Chapter VI

Concluding Remarks

In the past five chapters we have progressed through a variety of topics in an effort to gain understanding of Bonnor-Moffat-Boal unified field theory. Chapter 1 was a review of the subject of unified field theories of gravity and electromagnetism, chapter 2 a review of BMB theory with a discussion of the concept of duality and some interesting possible families of solutions. In chapter 3 we tried to extend the physical interpretation of all geometric quantities, which makes Einstein-Maxwell theory so aesthetically pleasing, to BMB. In chapter 4, perturbation expansions were made for a number of interesting cases, and with them an approximate BMB magnetic momopole was found. Lastly, in chapter 5, we discussed BMB radiation in the high frequency (geometric optics) limit.

Chapter 1 generally showed the troubles of unified field theories. Weyl's theory is an elegant gauge field extenesion of Riemannian geometry but it is difficult to find acceptable field equations. The five dimensional theories suffer from a lack of geometric interpretation of the fifth dimension, and they either generate unusual field equations or simply are equivalent to EMT. Einstein's nonsymmetric field seems like a logical generalization of general relativity, yet it has strange exact solutions and equations of motion.

In chapter 2, we saw that BMB theory is a modification of Einstein's nonsymmetric theory with improved equations of motion. Furthermore it is a generalization of EMT, not just of general relativity. It has an exact electric monopole solution , and may not have any magnetic monopole solution of similar form. To study the possibility of a magnetic monopole we looked at the concept of duality, and found that there is no simple way to formulate duality in a theory of Einsteinian nonsymmetric form.

Lastly, we pointed out several categories of possible solutions (eg, electromagnetic field on a flat background, or electromagnetic field with no electromagnetic potential) that should be investigated to establish some of the basic properties of BMB theory.

In chapter 3 we discovered some very basic problems in BMB theory in that a large portion of BMB's geometric quantities and properties is without satisfactory physical interpretation. There are two choices of connection, and an inner product which is not symmetric in its arguments. The (preferred) metric is not compatible with the (preferred) connection, so only very limited families of vectors have lengths and inner products preserved in parallel transport. This subsequently makes well-behaved frames difficult to construct. The hermitian formulation of the theory solves some of these problems, but creates others of its own. Lastly, Killing vector theory is considerably complicated by the nonsymmetric metric, and Killing vectorfields appear not to be associated with constants of motion in nonsymmetric field theories.

Most of chapter 4 consisted of perturbation expansions of the field equations for various generic cases. The lone application of these in the chapter was the derivation of an approximate magnetic monopole solution. Unfortunately, if there is a corresponding exact solution , it will likely be of a form difficult to find.

In chapter 5 we applied the weak field approximation of chapter 4 and the "geometric optics" high frequency approximation to radiation in five cases. For weak fields of arbitrary frequency on a flat background, and weak high frequency fields on a purely gravitational background we found fairly strong similarity between EMT and BMB radiation. However, for arbitrary strength high frequency waves in an exact solution, and for weak high frequency waves in a general back-

ground, we failed to find simple linearized equations, and probably should not attempt to draw any conclusions. Finally, for a one parameter weak high frequency case where we allowed curvature to diverge (within limits), we obtained simple results, but found them difficult to interpret.

The problems found in chapter 3 limit the present acceptability of BMB theory. Until a significant number of them are solved, the theory will not have solid physical interpretation of its mathematics. Without a reasonably complete physical interpretation the theory cannot be considered a serious competitor to EMT, and therefore it is the author's opinion that at present work in the theory should be directed toward resolving the interpretational problems.

References

1. Today most workers in general relativity appeal first to "the cosmic censorship hypothesis", the conjecture that normal processes in the universe, if they create singularities at all, create only singularities enclosed by event horizons (black hole surfaces), so that the singularities are not themselves visible in free space (this conjecture has yet to be proven), and second to the unification of general relativity and quantum mechanics, hoping that a renormalizable quantum field theory of gravitation will not have such singularities. There may in fact be some hope for freedom from singularities in the hermitian version of BMB theory, which we will come to later, and also in Kursonoglu's theory (Ref. 31).

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$$\hat{\phi}_a = \tfrac{1}{2}\,(\ln f)_{,a} \tag{R.1}$$

8. See Ref. 3, p. 295.

9. Ref. 5, p. 505 claims he did so in an appendix to Weyl's original paper.

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48. M. Wyman and H. Zassenhaus, Phys. Rev. 110, 228 (1958).

49. Most of this section can be found in any introductory text to general relativity.

50. Ref. 5, p. 164, or any standard vector analysis text.

51. Note the terminology convention used. Some unified field theorists call solutions of eq.(3.7) "autoparallel curves", and solutions of eq.(3.10) "geodesics". We are calling solutions of eq. (3.7) "geodesics", and those of eq.(3.10) "extremals of arc length".

52. The Lie derivative is one (generating tensors out of tensors) which is independent of the covariant derivative. It is defined in terms of some chosen vectorfield existing in the locale of interest.

$\mathcal{L}_{(k)}p^b$ happens to be (see Ref.53) the commutator of the two vectorfields k^a and p^b when they are considered as differential operators:

$$\mathcal{L}_{(k)}p^b = k^a p^b{}_{,a} - p^a k^b{}_{,a}. \tag{R.2}$$

53. C.W. Misner, K.S. Thorne, J.A. Wheeler, "Gravitation" (Freeman, San Francisco, 1970), p. 517.

54. In the exact solution (eq.(2.9),(2.10)) it can be shown that $\Gamma^a_{bc(1c)}$ generates different geodesics than Γ^a_{bc} (whose geodesics are the same as those of L^a_{bc}) from the same initial data (For example, in the exact solution, the geodesics of the two connections corresponding to a neutral test particle dropping radially inward from rest at some given radius are different four-space curves).

55. Work on unified field theories' equations of motion has not yet been carried to sufficiently high order to determine the paths of neutral particles.

56. As an example we can take the static, spherically symmetric solution, solve for any radial (timelike or null) geodesic, tangent k^a, and consider parallel propagation of a vector m^a which is null at a point p on the geodesic. It is not necessary to solve the parallel transport equations, we merely look at the derivative of m^a's magnitude at p :

$$k^c(g_{ab}m^a m^b)_{;c} = k^c g_{ab;c}m^a m^b + k^c g_{ab}m^a{}_{;c}m^b + k^c g_{ab}m^a m^b{}_{;c}. \tag{R.3}$$

Now, we know we can parallel transport m^a along the geodesic:

$$k^c m^a{}_{;c} = 0 \tag{R.4}$$

but that $g_{ab;c}$ is not zero. The term that is left is generally nonzero in the case described, so as soon as we move away from p, m^a's magnitude is nonzero, and it is no longer null.

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61. For a review of the geomtric optics of light in a curved background see Ref. 53, pp. 570 - 583. For geometric optics of the gravitational field itself see R.A. Isaacson, Ref. 60 and Phys. Rev. 166, 1272 (1968), and Ref. 53, pp. 964 - 973. It should be noted that geometric optics is not fully covariant. Isaacason, for example,

discusses the necessity of assuming the existence of well behaved "steady" coordinates in which the fields have the form we are considering. For recent work on more general cases in EMT see Refs. 62,63.

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64. There are entirely different methods of studying radiation. For example, a number of exact radiation field solutions are known and have been studied in general relativity (see J. Ehlers and W. Kundt in L. Witten, "Gravitation, an Introduction to Current Research" (Wiley, New York, 1962)), which are not asymptotically flat, and are therefore not easy to relate to truly physical situations. Another way to study radiation is to study the characteristic or singular surfaces of the theory. These submanifolds represent shock wave fronts (see, for example, Ref. 5, pp. 109 - 112, 284 - 289. For recent work in EMT see T. Tokuoka, J. Math. Phys. 15, 1972 (1974), and Refs. therin). In our methods we approximate physical radiation in the sense that the first order solutions we look at approximate an exact radiative solution satisfying the conditions of the physical situation (if such a solution exists).

65. This argument is due to M. Walker, private communication.

66. See Ref. 53. pp. 53 - 59.

67. To be specific, one asks that $\frac{1}{\varepsilon}$ be much less than the background's characteristic length scale of change, L, in the region of interest. We will take L as unity, to simplify expressions.

68. Some authors, eg. Isaacson, Ref. 60, assume a definite relation between λ and ε, eliminate one of the two, and work with the remaining one parameter expansions.

69. Isaacson, Refs. 60,61, finds an effective stress-energy-momentum tensor which is second order in amplitude for first order waves.

70. See Refs. 60, 61.

71. Or (more exotically), in the spirit of Isaacson (Refs. 60,61), $\overset{o}{g}_{(ab)}$ may be generated/modified by low frequency terms in higher order curvature actually generated by the high frequency disturbance. The differences between these two approaches do not appear in the leading order we are considering.

Appendix

Weak Fields in Minkowski Background, Second Order

Most of the second order quantities in the case of BMB weak fields in Minkowski space were written out in chapter 4, including $\overset{''}{\Gamma}{}^{a}_{bc}$ (starting at eq.(4.38), ending at eq.(4.51)). The remaining field equations are:

$$
\begin{aligned}
0 &= \overset{''}{\Gamma}{}^{c}_{[bc]} \\
&= \eta^{cd}(\overset{''}{g}_{[bc],d} - \overset{'}{g}_{[ec]}\overset{'}{\Gamma}{}^{e}_{(db)} - \overset{'}{g}_{[be]}\overset{'}{\Gamma}{}^{e}_{(cd)} - \overset{'}{g}_{(de)}\overset{'}{\Gamma}{}^{e}_{[bc]})
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
0 &= \overset{''}{R}_{(ab)} + \overset{''}{I}_{(ab)} \\
&= \overset{''}{\Gamma}{}^{c}_{(ab),c} - \tfrac{1}{2}(\overset{''}{\Gamma}{}^{c}_{(ac),b} + \overset{''}{\Gamma}{}^{c}_{(bc),a}) - \overset{'}{\Gamma}{}^{c}_{(ad)}\overset{'}{\Gamma}{}^{d}_{(cb)} \\
&+ \overset{'}{\Gamma}{}^{d}_{(cd)}\overset{'}{\Gamma}{}^{c}_{(ab)} - \overset{'}{\Gamma}{}^{c}_{[ad]}\overset{'}{\Gamma}{}^{d}_{[cb]} + \frac{8\pi}{k^2}(\eta^{cd}\overset{'}{g}_{[da]}\overset{'}{g}_{[cb]} \\
&- \tfrac{1}{4}\eta_{ab}\eta^{ce}\eta^{df}\overset{'}{g}_{[cd]}\overset{'}{g}_{[ef]})
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
0 &= {}_{[}(\overset{''}{R}_{[ab]} + \overset{''}{I}_{[ab]})_{,c]} \\
&= {}_{[}(\overset{''}{\Gamma}{}^{d}_{[ab],d} - \frac{8\pi}{k^2}\overset{''}{g}_{[ab]})_{,c]} .
\end{aligned}
\tag{A.3}
$$

As it is written, eq.(A.2) seems similar to Einstein's equation in EMT. Recalling that in chapter 4 we showed that first order EMT solutions on Minkowski background (Lorentz gauge) are solutions of BMB for the same case, using eq.(2.11):

$$g_{[ab]} = kF_{ab}$$

(F_{ab} the generalized electromagnetic field), and noting that:

$$\overset{''}{\Gamma}{}^{a}_{(bc)} = \overset{''}{\Gamma}{}^{a}_{(bc)emt} - \eta^{ad}\overset{'}{g}_{[ec]}\overset{'}{\Gamma}{}^{e}_{[db]} - \eta^{ad}\overset{'}{g}_{[be]}\overset{'}{\Gamma}{}^{e}_{[cd]} , \tag{A.4}$$

we see that eq.(A.2) can be written:

$$
\begin{aligned}
&\overset{''}{R}_{(ab)emt} - 8\pi\overset{''}{T}_{(ab)emt} - \eta^{cd}(\overset{'}{g}_{[eb]}\overset{'}{\Gamma}{}^{e}_{[da]} + \overset{'}{g}_{[ae]}\overset{'}{\Gamma}{}^{e}_{[db]})_{,c} \\
&- \tfrac{1}{2}\eta^{cd}(\overset{'}{g}_{[ec]}\overset{'}{\Gamma}{}^{e}_{[da]})_{,b} - \tfrac{1}{2}\eta^{cd}(\overset{'}{g}_{[ec]}\overset{'}{\Gamma}{}^{e}_{[db]})_{,a}
\end{aligned}
\tag{A.5}
$$

$$- \overset{\prime}{\Gamma}{}^{c}_{[ad]} \overset{\prime}{\Gamma}{}^{d}_{[cb]} = 0$$

where $\overset{\prime\prime}{T}_{(ab)emt}$ is the (second order) electromagentic stress-energy- momentum due to $\overset{\prime}{F}_{ab}$. The extra terms are all of order k^2, so this equation has the expected EMT limit.

Obviously the second order equations involve coupling to first order, so we cannot solve without using particular first order forms. We can, for example, look at the cases of only gravitational or only electromagnetic first order perturbation. These cases are possible since the perturbations are independent in first order. We know that all solutions of general relativity without electromagnetic field are also the purely gravitational solutions of BMB. From this, and knowledge of perturbation calculations in general relativity, we can predict that a gravitational perturbation to a geometry without electromagnetic background will not generate an electomagnetic perturbation in any order. This is supported by the BMB calculations on a purely gravitational background, and it can be seen here that if $\overset{\prime}{g}_{[ab]}$ is zero, then the equations in $\overset{\prime\prime}{g}_{[ab]}$ have trivial solution, and those in $\overset{\prime}{g}_{(ab)}$ are those of ordinary general relativity for the same circumstances. In the other case, ie, $\overset{\prime}{g}_{(ab)}$ zero, $\overset{\prime\prime}{g}_{[ab]}$ can again be zero, and $\overset{\prime}{g}_{(ab)}$ is governed by eq.(A.5).

Finally, if the first order solution is an electromagnetic plane wave, eq.(5,52),(5.54),(5.56):

$$\overset{\prime}{A}_a = \tfrac{1}{2}(\omega_a \exp (ik_c x^c) + cc)$$

$$\eta^{ab} k_a k_b = 0$$

$$\eta^{ab} k_a \omega_b = 0 ,$$

then:

$$\overset{\prime}{g}_{[ab]} = i\, \omega_{[a} k_{b]} \exp (ik_c x^c) + cc \tag{A.6}$$

$$\overset{\prime}{\Gamma}{}^{c}_{[ab]} = \exp(ik_e x^e)\,(-\eta^{cd}k_d\,\omega_{[a}k_{b]}) + cc. \tag{A.7}$$

$\overset{\prime}{g}_{[fc]}\overset{\prime}{\Gamma}{}^{c}_{[ab]}$ has a rather complicated form, but depends upon:

$$\omega_{[f}k_{c]}\,\eta^{cd}k_d\,\omega_{[a}k_{b]} \tag{A.8}$$

and similar expressions with $\overline{\omega}_a$'s replacing some or all the ω_a's. These are all easily seen to be zero, so:

$$\overset{\prime}{g}_{[fc]}\overset{\prime}{\Gamma}{}^{c}_{[ab]} = 0. \tag{A.9}$$

Similarly:

$$\overset{\prime}{\Gamma}{}^{c}_{[ad]}\overset{\prime}{\Gamma}{}^{d}_{[cb]} = 0\ , \tag{A.10}$$

and for first order plane electromagnetic perturbation on Minkowski background (in BMB), the second order gravitational field is governed by EMT's Einstein equation for the same situation.